Worksheet 23: Inner products and functional spaces

1–2. Prove that the following formulas do not define inner products on $\mathbb{R}^2$, by providing a property of the inner product that is violated:

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 - x_2 y_2, \tag{1}$$

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_2 - x_2 y_1. \tag{2}$$

Solutions: (1) If $\vec{x} = (0, 1)$, then $\langle \vec{x}, \vec{x} \rangle = -1 < 0$; property 4 is violated.

(2) $\langle \vec{y}, \vec{x} \rangle$ equals $-\langle \vec{x}, \vec{y} \rangle$ instead of $\langle \vec{x}, \vec{y} \rangle$; property 1 is violated. (Also, $\langle \vec{x}, \vec{x} \rangle = 0$ for any $\vec{x}$, which violates property 4.)

3. Lay, 6.7.19.

Solution: Put $\vec{v}_1 = (\sqrt{a}, \sqrt{b})$ and $\vec{v}_2 = (\sqrt{b}, \sqrt{a})$; then

$$\vec{v}_1 \cdot \vec{v}_2 = 2\sqrt{ab}, \ \ \|\vec{v}_1\| = \|\vec{v}_2\| = \sqrt{a+b}.$$

The Cauchy–Schwarz inequality gives $|\vec{v}_1 \cdot \vec{v}_2| \leq \|\vec{v}_1\| \cdot \|\vec{v}_2\|$, or

$$2\sqrt{ab} \leq a + b;$$

it remains to divide this by 2.

Problems 4–5 use the space $\mathbb{P}_2$ of polynomials of degree no more than 2, with the inner product

$$\langle f, g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1), \ \ f, g \in \mathbb{P}_2.$$

4. Prove that the system $\{t, t^2\}$ is orthogonal.

Solution: We calculate

$$\langle t, t^2 \rangle = -1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 = 0.$$

5. Find the orthogonal projection of the polynomial 1 onto the space $\mathrm{Span}\{t, t^2\}$.

Solution: We have

$$\begin{aligned}\langle 1, t\rangle &= 1\cdot(-1) + 1\cdot 0 + 1\cdot 1 = 0,\\ \langle t, t\rangle &= (-1)\cdot(-1) + 0\cdot 0 + 1\cdot 1 = 2,\\ \langle 1, t^2\rangle &= 1\cdot 1 + 1\cdot 0 + 1\cdot 1 = 2,\\ \langle t^2, t^2\rangle &= 1\cdot 1 + 0\cdot 0 + 1\cdot 1 = 2;\end{aligned}$$

the sought projection is given by

$$\frac{\langle 1, t\rangle}{\langle t, t\rangle} t + \frac{\langle 1, t^2\rangle}{\langle t^2, t^2\rangle} t^2 = t^2.$$

Problems 6–7 use the space $\mathbb{P}_2$, but now equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

6. Use Gram–Schmidt to find an orthogonal basis of $\mathbb{P}_2$, starting with the standard basis $\{1, t, t^2\}$. (You should first orthogonalize the system $\{1, t\}$ and return to t^2 if you have time left.)

Solution: We construct the sought basis $\{f_1, f_2, f_3\}$ step by step. Put $f_1 = 1$. Then,

$$\begin{aligned}\langle f_1, f_1\rangle &= \int_0^1 1\,dt = 1,\\ \langle t, f_1\rangle &= \int_0^1 t\,dt = \frac{1}{2},\\ \langle t^2, f_1\rangle &= \int_0^1 t^2\,dt = \frac{1}{3}.\end{aligned}$$

We then put

$$f_2 = t - \frac{\langle f_1, t\rangle}{\langle f_1, f_1\rangle} f_1 = t - 1/2;$$

we find

$$\begin{aligned}\langle f_2, f_2\rangle &= \int_0^1 (t - 1/2)^2\,dt = 1/12,\\ \langle f_2, t^2\rangle &= \int_0^1 t^2(t - 1/2)\,dt = 1/12;\end{aligned}$$

then, we put

$$f_3 = t^2 - \frac{\langle f_1, t^2 \rangle}{\langle f_1, f_1 \rangle} f_1 - \frac{\langle f_2, t^2 \rangle}{\langle f_2, f_2 \rangle} f_2 = t^2 - t + 1/6.$$

The resulting system is

$$\{1, t - 1/2, t^2 - t + 1/6\}.$$

7. Prove that the transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ that maps each polynomial $f(t)$ to the polynomial $f(1-t)$ is length preserving; i.e. for each $f \in \mathbb{P}_2$, $\|T(f)\| = \|f\|$.

Solution: We use the change of variables $s = 1 - t$:

$$\|T(f)\|^2 = \int_0^1 (f(1-t))^2 \, dt = \int_0^1 f(s)^2 \, ds = \|f\|^2.$$

Problems 8–9 use the space $C[-\pi, \pi]$ of all continuous functions $f : [-\pi, \pi] \to \mathbb{R}$, with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t) \, dt.$$

8. Show that $\sin t$ is orthogonal to $\cos t$. What is the length of $\sin t$? (Hint: use the double angle formulas. In case there is not enough time, set up the integrals, but do not compute them.)

Solution: To be posted on Wednesday.

9. Write a formula for the orthogonal projection of a function f onto the subspace of $C[-\pi, \pi]$ spanned by the constant function 1.

Solution: To be posted on Wednesday.