Worksheet 18: Diagonalization and diagonalizability

Given an $n \times n$ matrix A, here's what you need to do to diagonalize it:

(1) Compute the characteristic polynomial $P(\lambda) = \det(A - \lambda I)$. Its roots are the eigenvalues of A.

(2) If $P(\lambda)$ does not have n real roots, counting multiplicities (in other words, if it has some complex roots), then A is not diagonalizable.

(3) If for some eigenvalue λ, the dimension of the eigenspace $\mathrm{Nul}(A-\lambda I)$ is strictly less than the algebraic multiplicity of λ, then A is not diagonalizable.

(4) If neither (2) nor (3) hold, then A is diagonalizable. Find a basis for each eigenspace; combining these bases, you should get exactly n vectors $\vec{v}_1, \dots, \vec{v}_n$. Let D be the matrix whose diagonal elements are given by the eigenvalues corresponding to $\vec{v}_1, \dots, \vec{v}_n$ (in this order), and its offdiagonal elements are equal to zero. Define the square matrix P by its columns:

$$P = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}.$$

Then we have diagonalized A:

$$A = PDP^{-1}.$$

If you are able to diagonalize $A = PDP^{-1}$, then for every nonnegative integer k, the kth power of A can be computed by

$$A^k = PD^kP^{-1};$$

the matrix D^k is computed by taking the kth power of the diagonal elements of D.

1–3. Decide if the matrix A is diagonalizable. If it is, then diagonalize it (find D and P; you do not need to find P^{-1}).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix},$$
$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solutions: (1) The characteristic polynomial is $(1-\lambda)(\lambda^2 - 4\lambda + 3)$; the eigenvalues are 1 (multiplicity 2) and 3 (multiplicity 1). A basis for $\mathrm{Nul}(A - 1I)$ is $\{(1,0,0),(0,-1,1)\}$; a basis for $\mathrm{Nul}(A-3I)$ is $\{(0,1,1)\}$. The matrix A is diagonalizable, with

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(2) The characteristic polynomial is $(1-\lambda)(\lambda^2-4\lambda+5)$. Since $\lambda^2-4\lambda+5$ has only complex roots, A is not diagonalizable.

(3) The characteristic polynomial is $(1-\lambda)^2(2-\lambda)$; the eigenvalues are 1 (multiplicity 2) and 2 (multiplicity 1). A basis for $\mathrm{Nul}(A-1I)$ is $\{(1,0,0)\}$; since $\dim \mathrm{Nul}(A-1I) = 1$ is strictly less than the multiplicity of the eigenvalue 1, A is not diagonalizable.

4–6. Given the characteristic polynomial of the matrix A, decide whether (a) A is diagonalizable (b) A is not diagonalizable (c) A might or might not be diagonalizable, depending on the dimensions of eigenspaces:

$$P(\lambda) = (1-\lambda)(2-\lambda)^2(3-\lambda),$$
$$P(\lambda) = (1-\lambda)(2+\lambda^2)(3-\lambda),$$
$$P(\lambda) = (1-\lambda)(2-\lambda)(3-\lambda).$$

Answers: (4) Case (c); A is diagonalizable if and only if $\dim \mathrm{Nul}(A - 2I) = 2$. (5) Case (b), as the polynomial $2+\lambda^2$ has only complex roots. (6) Case (a), as all eigenvalues of A are real and distinct.

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Solution: We have

$$A^k = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4 - 3 \cdot 2^k & 12 \cdot 2^k - 12 \\ 1 - 2^k & 4 \cdot 2^k - 3 \end{bmatrix}.$$

8.* Define the sequence of **Fibonacci numbers** F_n by the recurrence relation

$$F_0 = 0, \;\; F_1 = 1; \;\; F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2.$$

The first several numbers in this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \ldots$$

(a) Define the vector $\vec{v}_n$ by

$$\vec{v}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

Prove that

$$\vec{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \vec{v}_n = A\vec{v}_{n-1}, \;\; n \geq 1, \;\; \text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(b) Diagonalize the matrix A. (If you do this right, you should get $\sqrt{5}$ somewhere.)

(c) Prove that $\vec{v}_n = A^n \vec{v}_0$; use this to derive **Binet's formula**:

$$F_n = \frac{\varphi^n - (\hat{\varphi})^n}{\sqrt{5}},$$

where

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

is the golden ratio and $\hat{\varphi} = 1 - \varphi$.

Solution: (a) We have $\vec{v}_0 = (F_0, F_1) = (0, 1)$. Next,

$$\vec{v}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} + F_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} = A\vec{v}_{n-1}.$$

(b) The characteristic polynomial is $\lambda^2 - \lambda - 1$. The eigenvalues are φ and $\hat{\varphi}$, defined above. We can write $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ \varphi & \hat{\varphi} \end{bmatrix}, \; D = \begin{bmatrix} \varphi & 0 \\ 0 & \hat{\varphi} \end{bmatrix}, \; P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\hat{\varphi} & 1 \\ \hat{\varphi} & -1 \end{bmatrix}.$$

(c) We have

$$\vec{v}_n = A\vec{v}_{n-1} = A^2\vec{v}_{n-2} = \cdots = A^n\vec{v}_0.$$

Next,

$$\begin{aligned} A^n\vec{v}_0 = PD^nP^{-1}\vec{v}_0 &= \begin{bmatrix} 1 & 1 \\ \varphi & \hat{\varphi} \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & \hat{\varphi}^n \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \varphi & \hat{\varphi} \end{bmatrix} \begin{bmatrix} \varphi^n \\ -\hat{\varphi}^n \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^n - \hat{\varphi}^n \\ \varphi^{n+1} - \hat{\varphi}^{n+1} \end{bmatrix}. \end{aligned}$$

By definition of $\vec{v}_n$, we get Binet's formula.

9.* (Nilpotent matrices and transformations) A square matrix A is called **nilpotent** if there exists a positive integer N such that $A^N = 0$.

(a) Prove that if A is nilpotent, then the only possible eigenvalue of A can be zero. (Hint: take a vector $\vec{x} \neq 0$ such that $A\vec{x} = \lambda\vec{x}$, and compute $A^N\vec{x}$.)

(b) Prove that if A is nilpotent and $A \neq 0$, then A is not diagonalizable. (Hint: assume that A is diagonalizable and use the formula $A = PDP^{-1}$; what is D?)

(c) A linear transformation $T : V \to V$ is called nilpotent if $T^N = 0$ for some N. (Here T^N means T composed with itself N times.) Prove that the transformation $T : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $T(f) = f'$ is nilpotent.

Solution: (a) Assume that λ is an eigenvalue of A. Then there exists $\vec{x} \neq 0$ such that $A\vec{x} = \lambda\vec{x}$. We can then compute $A^N\vec{x} = \lambda^n\vec{x}$ (similarly to what we did for A^2 last time). Since $A^N = 0$, we get $\lambda^n\vec{x} = \vec{0}$; but $\vec{x} \neq 0$, so $\lambda^n = 0$. It follows that $\lambda = 0$.

(b) We argue by contradiction. Assume that A is both nilpotent and diagonalizable; represent $A = PDP^{-1}$. By (a), the only eigenvalue of A is zero; since the diagonal entries of D are eigenvalues of A, we get $D = 0$. Then $A = P \cdot 0 \cdot P^{-1} = 0$.

(c) We claim that $T^4 = 0$. Indeed, $T^4 f$ is the fourth derivative of f; since f is a polynomial of degree no more than 3, we have $T^4 f = 0$.