Worksheet 17: Characteristic polynomial and a glimpse of diagonalization

1. Find the characteristic polynomial and the real eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Answer: The characteristic polynomial is $\lambda^2 - 5\lambda + 6$; the eigenvalues are 2 and 3.

2. Find the characteristic polynomial and the real eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Recalling that A is the standard matrix of a 90 degree rotation, find a geometric interpretation of the answer.

3. Lay, 5.2.15.

Answer: 4 (multiplicity 1), 3 (multiplicity 2), 1 (multiplicity 1). The characteristic polynomial is $(4-\lambda)(3-\lambda)^2(1-\lambda)$.

4. For the matrix A in problem 1, let λ_1, λ_2 be the eigenvalues. Find an eigenvector $\vec{v}_1$ for the eigenvalue λ_1 and an eigenvector $\vec{v}_2$ for the eigenvalue λ_2. Consider the matrix

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

and the diagonal 2×2 matrix D whose diagonal entries are λ_1 and λ_2. Show that

$$PD = AP.$$

Show that P is invertible and prove that

$$A = PDP^{-1}.$$

Solution: From problem 1, we know that

$$\lambda_1 = 2,\ \lambda_2 = 3.$$

Next, we row reduce $A - 2I$ and $A - 3I$:

$$A - 2I = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix};$$
$$A - 3I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}.$$

We can put

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So,

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},\ D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

P is invertible since $\det P \neq 0$. Next,

$$PD = \begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix} = AP.$$

Multiplying this by P^{-1} to the right, we get $A = PDP^{-1}$. Note that no matter what $\vec{v}_1$ and $\vec{v}_2$ are, we have

$$PD = \begin{bmatrix} 2\vec{v}_1 & 3\vec{v}_2 \end{bmatrix},\ AP = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 \end{bmatrix}.$$

So, the identity $PD = AP$ follows from the eigenvector identities $A\vec{v}_1 = 2\vec{v}_1$, $A\vec{v}_2 = 3\vec{v}_2$.

5. Lay, 5.2.19.

Solution: Substitute $\lambda = 0$ into the equation

$$\det(A - \lambda I) = (\lambda_1 - \lambda)\dots(\lambda_n - \lambda).$$