Worksheet 12: Subspaces and bases

1–4. For each of the following sets, either prove that it is not a subspace of $\mathbb{R}^n$, or represent it as $\operatorname{Col} A$ or $\operatorname{Nul} A$ for some matrix A:

$$\{(a,b,c,d) \mid a-2b=4c,\ 2a=c+3d\}, \tag{1}$$
$$\{(a,b,c) \mid a+b=c+2\}, \tag{2}$$
$$\{(a-b,a+b,b+1) \mid a,b \in \mathbb{R}\}, \tag{3}$$
$$\{(-a+2b,a-2b,3a-6b) \mid a,b \in \mathbb{R}\}. \tag{4}$$

Answers: (1) This is $\operatorname{Nul} A$, for

$$A = \begin{bmatrix} 1 & -2 & 4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}.$$

(2) Not a subspace, as does not contain the zero vector.

(3) Not a subspace, as does not contain the zero vector. Indeed, if $(a-b, a+b, b+1) = (0,0)$, then $a-b=a+b=0$; thus, $a=b=0$, which contradicts that $b+1=0$.

(4) This is $\operatorname{Col} A$, for

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}.$$

5–8. Determine which of these sets form a basis of $\mathbb{R}^3$. For those sets which are not bases, state whether they do not span $\mathbb{R}^3$, are not linearly independent, or both:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}; \tag{5}$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} \right\}; \tag{6}$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}; \tag{7}$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}. \tag{8}$$

Answers: (5) Not a basis — linearly independent, but do not span $\mathbb{R}^3$ (6) Not a basis — linearly dependent and do not span $\mathbb{R}^3$ (7) Basis (8) Not a basis — span $\mathbb{R}^3$, but linearly dependent.

9. Prove that every basis of $\mathbb{R}^3$ consists of 3 vectors.

Solution: Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis of $\mathbb{R}^3$. Then if $n > 3$, these vectors cannot be linearly independent; if $n < 3$, they cannot span $\mathbb{R}^3$. Therefore, $n = 3$.

10. Lay, 4.3.13.

Answer: Basis for $\operatorname{Nul} A$: $(-6, -5/2, 1, 0), (-5, -3/2, 0, 1)$. Basis for $\operatorname{Col} A$: $(-2, 2, -3), (4, -6, 8)$.

11. Does there exist a subspace W of $\mathbb{R}^3$ such that the vectors from problem 5 form a basis of W? What about the vectors from problem 8?

Solution: The vectors in problem 5 are linearly independent and form a basis of the subspace spanned by these vectors. The vectors in problem 8 are linearly dependent and cannot form a basis of anything.

100.* Let A be an $m \times n$ matrix, and $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by the formula $T(\vec{x}) = A\vec{x}$.

(a) Let X be a subspace of $\mathbb{R}^n$. Prove that the set

$$T(X) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$$

is a subspace of $\mathbb{R}^m$.

(b) Let Y be a subspace of $\mathbb{R}^m$. Prove that the set

$$T^{-1}(Y) = \{\vec{x} \mid T(\vec{x}) \in Y\}$$

is a subspace of $\mathbb{R}^n$. (Caution: T need not be invertible for $T^{-1}(Y)$ to be well defined!)

Solution: We only verify that $T(X)$ and $T^{-1}(Y)$ are closed under addition; the rest is left to the reader.

(a) Let $\vec{y}_1, \vec{y}_2 \in T(X)$. Then there exist $\vec{x}_1, \vec{x}_2 \in X$ such that $\vec{y}_1 = T(\vec{x}_1), \vec{y}_2 = T(\vec{x}_2)$. Since X is a subspace, $\vec{x}_1 + \vec{x}_2 \in X$; since T is linear, $\vec{y}_1 + \vec{y}_2 = T(\vec{x}_1 + \vec{x}_2) \in T(X)$.

(b) Let $\vec{x}_1, \vec{x}_2 \in T^{-1}(Y)$. Then $T(\vec{x}_1), T(\vec{x}_2) \in Y$. Since T is linear, $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$; since Y is a subspace, $T(\vec{x}_1) + T(\vec{x}_2) \in Y$. Therefore, $T(\vec{x}_1 + \vec{x}_2) \in Y$; it follows that $\vec{x}_1 + \vec{x}_2 \in T^{-1}(Y)$.