Worksheet 11: Subspaces

We will consider the following vector spaces:

- $\mathbb{R}^n$, the spaces we studied before;
- $\mathbb{P}_n$, the space of all polynomials in one variable of degree $\leq n$;
- $\mathbb{P}$, the space of all polynomials.

1–4. Are the following sets subspaces of $\mathbb{R}^2$?
(1) The line passing through $(0, 1)$ and $(1, 0)$.
(2) The line passing through $(0, 1)$ and $(0, -1)$.
(3) The disc of radius 1 centered at zero.
(4) The set of points (x_1, x_2) such that $x_1 x_2 = 0$.

Answers: (1) No, as it does not contain the zero vector (2) Yes (3) No, as $(1, 0)$ lies in the disc, but $2(1, 0)$ does not, thus violating property (c) of the definition of a subspace (4) No, as $(1, 0)$ and $(0, 1)$ lie in this set, but their sum does not.

5. Find a vector $\vec{v}$ such that the set from problem 2 is equal to $\mathrm{Span}(\vec{v})$.
Answer: One possibility is $\vec{v} = (0, 1)$.

6. Represent the set

$$\{(a - b, b - c, c - a) \mid a, b, c \in \mathbb{R}\} \subset \mathbb{R}^3$$

as $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for some vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Prove that this set is a subspace of $\mathbb{R}^3$.

Solution: We have

$$\begin{bmatrix} a - b \\ b - c \\ c - a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix};$$

therefore, the set in question is spanned by $\vec{v}_1 = (1, 0, -1), \vec{v}_2 = (-1, 1, 0), \vec{v}_3 = (0, -1, 1)$.

7. Show that the set

$$S = \{f(x) \in \mathbb{P} \mid f(7) = 0\}$$

is a subspace of $\mathbb{P}$.

Solution: We verify the properties in the definition of a subspace:

- The zero polynomial is in S, as its value at any point is equal to zero.
- Let $f, g \in S$. Then $(f + g)(7) = f(7) + g(7) = 0$; therefore, $f + g \in S$.
- Let $f \in S$ and $c \in \mathbb{R}$. Then $(cf)(7) = cf(7) = 0$; therefore, $cf \in S$.

8. Using the method of Problem 6, show that the set

$$\{a + (a + b)t^2 \mid a, b \in \mathbb{R}\}$$

is a subspace of $\mathbb{P}_2$.

Solution: We have

$$a + (a + b)t^2 = a(1 + t^2) + bt^2.$$

Therefore, the set in question is spanned by $\{1 + t^2, t^2\}$. It follows that it is a subspace of $\mathbb{P}_2$.

9.* Lay, 4.1.33.

Solution: We verify the defining properties of a subspace for $H + K$:

- $\vec{0} \in H + K$, as we can represent $\vec{0} = \vec{0} + \vec{0}$ with $\vec{0} \in H$ and $\vec{0} \in K$.
- Assume that $\vec{w}_1, \vec{w}_2 \in H + K$. Then there exist $u_1, u_2 \in H$ and $v_1, v_2 \in K$ such that $\vec{w}_1 = \vec{u}_1 + \vec{v}_1$ and $\vec{w}_2 = \vec{u}_2 + \vec{v}_2$. Then we represent $\vec{w}_1 + \vec{w}_2 = (\vec{u}_1 + \vec{u}_2) + (\vec{v}_1 + \vec{v}_2)$ with $\vec{u}_1 + \vec{u}_2 \in H$ and $\vec{v}_1 + \vec{v}_2 \in K$; therefore, $\vec{w}_1 + \vec{w}_2 \in H + K$.
- Assume that $\vec{w} \in H + K$ and $c \in \mathbb{R}$. Then there exist $\vec{u} \in H$ and $\vec{v} \in K$ such that $\vec{w} = \vec{u} + \vec{v}$. Then we represent $c\vec{w} = c\vec{u} + c\vec{v}$ with $c\vec{u} \in H$ and $c\vec{v} \in K$; therefore, $c\vec{w} \in H + K$.