Math 54-1 Your name: Key

Quiz 2, July 2, 2010

Please write your name **on each sheet**. Show your work clearly and in order, including intermediate steps in the solutions and the final answer.

1. (5 pt) Describe the set of all solutions of the vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

in parametric vector form. That is, find the general form of the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

The vector eqn is equivalent to the system of linear equations with the augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} \boxed{1} & 2 & 0 & 2 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} \boxed{1} & 0 & -2 & 2 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

General solution: $X_1 = 2 + 2X_3$
$X_2 = -X_3$
X_3 free.

So, $\vec{X} = \begin{bmatrix} 2 + 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. So,

the solution set is $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \vec{v} \;\middle|\; \vec{v} \in \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \right\}$.

2. (5 pt) Consider the vectors

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

(a) Prove that the vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ form a linearly dependent set. Find a linear dependence relation.

(b) Use the linear dependence relation found in (a) to express one of the vectors above as a linear combination of the other two.

(c) ~~(c)~~ Do the vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ span the whole $\mathbb{R}^2$? Explain. (A picture with no explanations would not suffice.)

(a) $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \mid 0] = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 4 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1}$

$\longrightarrow \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow \frac{-R_2}{2}} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \longrightarrow$

$\xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} \boxed{1} & 0 & 3 & 0 \\ 0 & \boxed{1} & -1 & 0 \end{bmatrix}$; general solution to the corresponding system of linear equs is: $x_1 = -3x_3$, $x_2 = x_3$, x_3 free

A nonzero solution for this system is given by $x_3 = 1$, $x_1 = -3$, $x_2 = 1$, and produces the linear dependence relation

$$\boxed{-3\vec{a}_1 + \vec{a}_2 + \vec{a}_3 = 0}$$

(b) We rewrite the relation above as $\vec{a}_1 = \dfrac{\vec{a}_2 + \vec{a}_3}{2} \in$
$\in \text{Span}\{\vec{a}_2, \vec{a}_3\}$.

(c) They do, as $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ is row equivalent to $\begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -1 \end{bmatrix}$, with pivot in each row.