Math 54, second midterm information and review

July 29, 2010

1 General information

The second midterm will take place on Friday, July 30, from 8–10 AM in room 2 Evans. The exam itself will start at 8:10, but I ask you to come at 8 so that I could hand out the exams and everybody would start at the same time. There are no calculators and no materials allowed, except for one two-sided 5”×9” sheet of hand-written notes. Do not bring your own paper — I will provide extra sheets if needed. The midterm will cover Lay, Chapters 4, 5, and sections 6.1–6.3. As before, 60% will be devoted to computational problems and the rest to theoretical problems.

2 Sample computational problems

1–2. For each of the following subspaces W:
(a) Represent W as either $\operatorname{Col} A$ or $\operatorname{Nul} A$ for some matrix A.
(b) Find a basis for W.
(c) State the dimension of W.
(d)–(f) Repeat (a)–(c) for the orthogonal complement $W^\perp$.

$$W = \{(a-b, b-c, c-a) \mid a, b, c \in \mathbb{R}\}; \tag{1}$$

$$W = \{(a, b, c) \mid a+b=0,\ b+c=0,\ a=c\}. \tag{2}$$

3–5. For each of the following matrices:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \tag{3}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{4}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{5}$$

(a) Find the characteristic polynomial of A.

(b) Find the real eigenvalues of A and state their algebraic multiplicities.
(c) Find the basis of each eigenspace of A.
(d) Is A diagonalizable? If it is diagonalizable, represent it as PDP^{-1}, with D diagonal and P invertible.
(e) If A is diagonalizable, find the general formula for its power A^k.

6. Given the transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by the formula

$$T(f) = (t+1)f', \ f \in \mathbb{P}_2,$$

the basis $\mathcal{C} = \{1, t, t^2\}$ of $\mathbb{P}_2$ and the system $\mathcal{B} = \{1, 1+t, (1+t)^2\}$ in $\mathbb{P}_2$,
(a) Find the matrix A of T in the basis $\mathcal{C}$.
(b) Is T 1-to-1? Is it onto?
(c) Find a basis for the kernel of T and for its range. (For that, find the bases for $\operatorname{Nul} A$ and $\operatorname{Col} A$ and translate these from $\mathcal{C}$-coordinate vectors to polynomials.)
(d) Use $\mathcal{C}$-coordinate vectors to prove that $\mathcal{B}$ is a basis of $\mathbb{P}_2$.
(e) Find the change of coordinate matrices $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ and $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}}$.
(f) Find the $\mathcal{C}$-coordinate vector of the polynomial $f = 1 + 2t + 3t^2$. Use part (e) to find the $\mathcal{B}$-coordinate vector of f.
(g) Use parts (a) and (e) to find the matrix of T in the basis $\mathcal{B}$.

7. Given $\vec{u} = (1, t, 2)$, $\vec{v}_1 = (1, 1, 1)$, $\vec{v}_2 = (1, 1, -2)$, verify that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is an orthogonal system. Let $V = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$; use the orthogonal projection formula to:
(a) Find the orthogonal projection of $\vec{u}$ onto V (depending on t).
(b) Find the distance from $\vec{u}$ to V (depending on t).
(c) Find for which t the vector $\vec{u}$ lies in V.
(d) For t such that $\vec{u} \in V$, find the coordinates of $\vec{u}$ in the basis $\mathcal{B}$.

8. Assuming that A is a 4×8 matrix with $\dim \operatorname{Nul} A = k$,
(a) Find all possible values of k.
(b) Find the rank of A (depending on k).
(c) Define the linear transformation $T : \mathbb{R}^8 \to \mathbb{R}^4$ by the formula $T(\vec{x}) = A\vec{x}$. For which k is T onto? For which k is it 1-to-1?
(d) Answer part (c) for A^T in place of A.

3 Sample theoretical problems

1. Let A be a matrix. Use the Rank Theorem to prove that the solution to $A\vec{x} = 0$ is unique if and only if the equation $A^T\vec{y} = \vec{b}$ has a solution for each $\vec{b}$.
2. Assume that the matrix A satisfies the equation

$$A^3 = A.$$

What are the possible eigenvalues of A?

3. Let A be a 2×2 matrix. Define the trace $\operatorname{tr} A$ as the sum of its diagonal entries. Prove that the characteristic polynomial of A is

$$P(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A.$$

Use the quadratic formula to find when A has two distinct real eigenvalues. Use it again to prove that if A has two real eigenvalues $\lambda_1 \neq \lambda_2$, then $\lambda_1 + \lambda_2 = \operatorname{tr} A$ and $\lambda_1 \cdot \lambda_2 = \det A$.

4. Give an example of a 2×2 orthogonal matrix A such that $A^2 = -I$.

5. Assume that A is a matrix such that $A^2 = -I$. Prove that A has no real eigenvalues.

6. Let T and S be two linear transformations such that the composition $T \circ S$ is well defined. (Recall that $T \circ S$ is defined by the formula $(T \circ S)(\vec{v}) = T(S(\vec{v}))$.) Prove that the kernel of S is contained in the kernel of $T \circ S$ and the range of $T \circ S$ is contained in the range of T.

7. Let V be a vector space and $T : V \to V$ be a linear transformation such that $T^2 = 0$. (Here 0 is the zero transformation $V \to V$, mapping every vector to the zero vector.) Prove that the range of T is contained in its kernel.

4 Answers to computational problems

1. (a) $W = \operatorname{Col} A$, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

(b) $\{(1,0,-1),(-1,1,0)\}$ (c) 2 (d) $W^\perp = \operatorname{Nul} A^T$, where

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) $\{(1,1,1)\}$ (c) 1.

2. (a) $W = \operatorname{Nul} A$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

(b) $\{(1,-1,1)\}$ (c) 1 (d) $W^\perp = \operatorname{Col} A^T$, where

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

(e) $\{(1,1,0),(0,1,1)\}$ (f) 2.

3. (a) $\lambda^2 - \lambda - 2$ (b) $-1, 2$ (both multiplicity 1) (c) For $\lambda = -1$: $\{(-2,1)\}$; for $\lambda = 2$: $\{(1,1)\}$ (d) Yes;

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \; P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

(e)

$$A^k = \frac{1}{3}\begin{bmatrix} 2(-1)^k + 2^k & 2(-1)^{k+1} + 2^{k+1} \\ (-1)^{k+1} + 2^k & (-1)^k + 2^{k+1} \end{bmatrix}.$$

4. (a) $(\lambda - 1)^2$ (b) 1 (multiplicity 2) (c) For $\lambda = 1$: $\{(0,1)\}$ (d) No.

5. (a) $(1-\lambda)^2(2-\lambda)$ (b) 1 (multiplicity 2), 2 (multiplicity 1) (c) For $\lambda = 1$: $\{(0,1,0)\}$; for $\lambda = 2$: $\{(0,0,1)\}$ (d) No.

6. (a)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Not 1-to-1, not onto (c) Basis for the kernel: $\{1\}$; basis for the range: $\{1+t, 2t+2t^2\}$ (e)

$$\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(f)

$$[f]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ [f]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}.$$

(g) Note also that $1, 1+t, (1+t)^2$ are eigenvectors of T.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

7. (a) $(1+t, 1+t, 4)/2$ (b) $|t-1|/sqrt2$ (c) $t=1$ (d) $(4/3, -1/3)$.

8. (a) $k = 4, 5, 6, 7, 8$ (b) $8-k$ (c) Onto for $k=4$, never 1-to-1 (d) 1-to-1 for $k=4$, never onto.

5 Hints and answers for theoretical problems

1. Assume that A has dimension $m \times n$. Recall that the ranks of A and A^T are equal. Now, formulate both two statements of the problem in terms of the rank.

2. Every eigenvalue λ solves the equation $\lambda^3 = \lambda$; therefore, the possible eigenvalues are $0, 1, -1$.

3. A has two distinct real eigenvalues iff $(\operatorname{tr} A)^2 > 4 \det A$. For the last part, you can write $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$, expand this expression and compare the coefficients at λ and λ^2 with $-\operatorname{tr} A$ and $\det A$.

4. Take the standard matrix of a 90 degree rotation (in either direction).

5. Each eigenvalue λ solves the equation $\lambda^2 = -1$.

6. First, assume that $\vec{v}$ lies in the kernel of S. Then $S(\vec{v}) = \vec{0}$ and $(T \circ S)(\vec{v}) = T(S(\vec{v})) = T(\vec{0}) = \vec{0}$; therefore, $\vec{v}$ lies in the kernel of $T \circ S$. We have proven that the kernel of S is contained in the kernel of $T \circ S$.

Now, assume that $\vec{v}$ lies in the range of $T \circ S$. Then there exists $\vec{w}$ such that $\vec{v} = T(S(\vec{w}))$. If we put $\vec{u} = S(\vec{w})$, then $\vec{v} = T(\vec{u})$; therefore, $\vec{v}$ lies in the range of T. We have proven that the range of $T \circ S$ is contained in the range of T.

7. Take $\vec{w}$ in the range of T. Then there exists $\vec{v}$ such that $\vec{w} = T(\vec{v})$. However, since $T^2 = 0$, we have $\vec{0} = T^2\vec{v} = T(T(\vec{v})) = T(\vec{w})$. It follows that $\vec{w}$ lies in the kernel of T.