

Math 54, midterm 2

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Name: _____ SID: _____

Problem 1: _____ / 10

Problem 2: _____ / 10

Problem 3: _____ / 10

Problem 4: _____ / 10

Problem 5: _____ / 10

Total: _____ / 50

- Write your solutions in the space provided. Do not use your own paper. I can give you extra paper if needed. Indicate clearly where your answer is.
- Explain your solutions as clearly as possible. This will help me find what you did right and what you did wrong, and award partial credit if possible.
- Justify all your steps. (Problem 4(a) is exempt from this rule.) A correct answer with no justification will be given 0 points. Pictures without explanations are not counted as justification. You may cite a theorem from the book by stating what it says.
- No calculators or notes are allowed on the exam, except for a single two-sided 5" × 9" sheet of hand-written notes. Cheating will result in academic and/or disciplinary action. Please turn off cellphones and other electronic devices.

1. Consider the linear transformation $T : \mathbb{P}_2 \rightarrow \mathbb{R}^3$ given by the formula

$$T(f) = \begin{bmatrix} f(-1) \\ f'(0) \\ f(1) \end{bmatrix}, \quad f \in \mathbb{P}_2.$$

(a) Find the matrix A of the transformation T relative to the basis $\mathcal{B} = \{1, t, t^2\}$ of \mathbb{P}_2 and the standard basis $\mathcal{C} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of \mathbb{R}^3 .

(b) Find the rank of the matrix A from part (a) and the dimensions of the kernel of T and the range of T .

2. (a) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

and find a formula for A^k , where k is a nonnegative integer. Your final answer should be written as a single matrix (4 numbers between the square brackets) with the entries depending only on k .

(b) Is the following matrix diagonalizable? Explain. (Do not diagonalize.)

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ 5 & 0 & 4 \end{bmatrix}.$$

3. (a) Express the subspace

$$W = \{(a - b + c, b, a + b + c) \mid a, b, c \in \mathbb{R}\}$$

of \mathbb{R}^3 as either $\text{Col } A$ or $\text{Nul } A$ for some matrix A . Find a basis for W and a basis for W^\perp .

(b) Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

in \mathbb{R}^3 . Prove that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal system and find the orthogonal projection of \vec{v} onto $\text{Span}\{\vec{u}_1, \vec{u}_2\}$.

4. (a) Circle the correct answer for each of the following true/false questions. No justification is required. For each question, you get 1 point if you answer it correctly, 0 points if you choose not to answer, and -1 point if you answer it incorrectly. (If your total score for this part of the problem is negative, it will be changed to zero.)

(1) If the matrix A is invertible, then 0 is not an eigenvalue of A .

True False

(2) The product of two orthogonal matrices of the same size is an orthogonal matrix.

True False

(3) If a matrix A has characteristic polynomial $-\lambda^3 - \lambda$, then it is diagonalizable.

True False

(4) If A is row equivalent to B , then the eigenvalues of A and B are the same.

True False

(5) If A is row equivalent to B , then $\text{Nul } A = \text{Nul } B$.

True False

(b) Let A be a square matrix such that $A^3 = A^5$. What are the possible eigenvalues of A ? Explain.

5. Solve **one** of the following two problems. Mark which one you want graded.

(a) Assume that A is a 2×2 orthogonal matrix and $\det A = -1$. Prove that $A^2 = I$. Your proof has to be purely algebraic, starting with the definition of an orthogonal matrix and not appealing directly to the geometric properties of orthogonality that we studied in class.

(b) Assume that U, V, W are vector spaces and $T : V \rightarrow W$, $S : U \rightarrow V$ are linear transformations such that $T \circ S = 0$. (In other words, $T(S(\vec{u})) = \vec{0}$ for each $\vec{u} \in U$.) Prove that the kernel of T contains the range of S .