Math 54, midterm 1

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Name: Key SID: —

Problem 1: — / 10

Problem 2: — / 10

Problem 3: — / 10

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- Write your solutions in the space provided. Do not use your own paper. I can give you extra paper if needed. Indicate clearly where your answer is.
- Explain your solutions as clearly as possible. This will help me find what you did right and what you did wrong, and award partial credit if possible.
- Justify all your steps. (Problem 3 is exempt from this rule.) A correct answer with no justification will be given 0 points. Pictures without explanations are not counted as justification. You may cite a theorem from the book by stating what it says.
- No calculators or notes are allowed on the exam, except for a single two-sided 5"×9" sheet of hand-written notes. Cheating will result in academic and/or disciplinary action. Please turn off cellphones and other electronic devices.

1. (a) Find the standard matrix of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula $T(x_1, x_2, x_3) = (x_1 + 2x_2, x_2 + 2x_3)$.

We define $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then the standard matrix of T is

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)] =$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

$T(\vec{e}_1) = T(1,0,0) =$
$= (1,0)$;
$T(\vec{e}_2) = T(0,1,0) =$
$= (2,1)$
$T(\vec{e}_3) = T(0,0,1) =$
$= (0,2)$

(b) Describe the solution set of the equation $T(\vec{x}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in parametric vector form.

$T(\vec{x}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ means $A \cdot \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$; solving:

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}}_{\text{augmented matrix}} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} \boxed{1} & 0 & -4 & -5 \\ 0 & \boxed{1} & 2 & 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= -5 + 4x_3 \\ x_2 &= 3 - 2x_3 \\ x_3 &\text{ free} \end{aligned}}$$

$$\boxed{\vec{x} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}}$$

2. (a) Let A be a 3×3 matrix such that $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$. Solve the equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. There is only one solution

$$\vec{x} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 17 \end{bmatrix}.$$

(b) Find a linear dependence relation between the vectors

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Write $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \cancel{\#} 0] = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 1 & 6 & 0 \end{bmatrix} \longrightarrow$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R_2 \leftarrow R_2 \cdot (-1)} \begin{bmatrix} \boxed{2} & 0 & 3 & 0 \\ 0 & \boxed{1} & 0 & 0 \end{bmatrix};$$ the general solution is $x_1 = -\frac{3}{2} x_3$, $x_2 = 0$, x_3 free

Pick a nonzero solution: $x_3 = 1$, $x_2 = 0$, $x_1 = -\frac{3}{2}$. This leads to the relation $\boxed{-\frac{3}{2}\vec{a}_1 + \vec{a}_3 = 0.}$

3. Circle the correct answer for each of the following true/false or multiple choice questions, or write in the correct answer in the blank space provided. No justification is required. For each question, you get 1 point if you answer it correctly, 0 points if you choose not to answer, and -1 point if you answer it incorrectly. (If your total score for the problem is negative, it will be changed to zero.)

(a) For each matrix A, $\det(A^T A) \geq 0$.

True   False $\det(A^T A) = \det A^T \cdot \det A = \det A \cdot \det A = (\det A)^2 \geq 0.$

(b) For all square matrices A and B of the same size, if $AB = 0$, then $A = 0$ or $B = 0$.

True   False   Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(c) For all square matrices A and B of the same size, $\det(A^2 - B^2) = \det(A + B) \cdot \det(A - B)$.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A^2 - B^2 = 0$,

True   False   but $\det(A+B) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, $\det(A-B) = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2$

(d) If the equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$, then A is square and $\det A \neq 0$.

True   False   Take $A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{bmatrix}$; pivot in every ~~column~~ row, but not square

(e) If AB is well-defined and invertible and A is a square matrix, then B is invertible.

AB invertible $\to$ AB square $\to$ B square.

True   False $0 \neq \det(AB) = \det A \cdot \det B \Rightarrow \det B \neq 0 \Rightarrow B$ is invertible

(f) If A has size $m \times n$ and B has size $\underline{n} \times \underline{k}$, then AB is well-defined and has size $\underline{m} \times k$.

(g) If a linear transformation $T : \mathbb{R}^a \to \mathbb{R}^b$ is onto, then:

$a \leq b$ $a \geq b$ The standard matrix of T has size $b \times a$ and has to have a pivot in each row.

(h) If the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^4$ form a linearly dependent set, then each of these vectors is a linear combination of the other two.

True   False   Take $\vec{a}_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}$, then $\vec{a}_2 \notin \text{Span}\{\vec{a}_1, \vec{a}_3\}$

(i) For each matrix A, if the equation $A\vec{x} = \vec{0}$ has unique solution, then for each $\vec{b}$, the number of solutions to the equation $A\vec{x} = \vec{b}$ is either 0 or 1.

If $A\vec{x} = \vec{b}$ is inconsistent, then we have 0 solutions

True   False   If $A\vec{x} = \vec{b}$ is consistent, then we have 1 solution (Th. 1.5.6)

(j) If A is a square matrix and $A^T A = I$, then $AA^T = I$.

True   False   Put $B = A^T$. Then $BA = I$; by a problem we did in class, since A is square, $B = A^{-1}$ and $AA^T = AB = \cancel{I} A \cdot A^{-1} = I$.

4. (a) Find all $t \in \mathbb{R}$ such that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + t \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ has linearly dependent columns.

A has lin. dep. cols $\overset{\text{IMT}}{\iff}$ ~~det~~ A is not invertible $\iff$

$\iff \det A = 0$, but $\det A = \det \begin{bmatrix} 1+t & 4t \\ 1 & 1+t \end{bmatrix} =$

$= (1+t)^2 - 4t = (1-t)^2$. So,

A has linearly dependent columns iff $\boxed{t = 1}$

(b) Find $\det \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 5 \\ 1 & 3 & 0 & 5 \end{bmatrix} \underset{\text{down col 3}}{\overset{\text{cofactor expand}}{=}} -2 \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 3 & 5 \end{bmatrix} =$

$\overset{R_3 \leftarrow R_3 - R_1}{=} -2 \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} \underset{\text{down col 1}}{\overset{\text{cofactor expand}}{=}} -2 \det \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} =$

$= -2(4 \cdot 2 - 1 \cdot 5) = -6.$

5. Solve **one** of the following two problems. Mark which one you want graded.

(a) Assume that A and B are square matrices of the same size and $A^2 = B^2 = (AB)^2 = I$. Show that $AB = BA$.

$I = (AB)^2 = AB \cdot AB = ABAB$

Multiply by A to the left:

$A = A^2 BAB = I\, BAB = BAB$

Multiply by B to the right:

$AB = BAB^2 = BAI = BA.$

(b) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^n$. Prove that $\mathrm{Span}\{\vec{a}_1, \vec{a}_2\} \subset \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Here the sign '$\subset$' means 'is contained in'. Also, prove that if $\vec{a}_3 \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$, then $\mathrm{Span}\{\vec{a}_1, \vec{a}_2\} = \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

1 Let $\vec{u} \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$. Then $\vec{u} = c_1\vec{a}_1 + c_2\vec{a}_2$ for some $c_1, c_2 \in \mathbb{R}$. Represent $\vec{u} = c_1\vec{a}_1 + c_2\vec{a}_2 + 0\,\vec{a}_3$; this shows that $\vec{u} \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

We proved that $\mathrm{Span}\{\vec{a}_1, \vec{a}_2\} \subset \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

2 Since $\vec{a}_3 \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$, $\vec{a}_3 = d_1\vec{a}_1 + d_2\vec{a}_2$ for some $d_1, d_2 \in \mathbb{R}$. Now, take $\vec{u} \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

Then $\vec{u} = c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

We then write $\vec{u} = c_1\vec{a}_1 + c_2\vec{a}_2 + c_3(d_1\vec{a}_1 + d_2\vec{a}_2) =$

$= (c_1 + c_3 d_1)\vec{a}_1 + (c_2 + c_3 d_2)\vec{a}_2 \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$.

We proved that $\mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} \subset \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$; together with 1, this shows that $\mathrm{Span}\{\vec{a}_1, \vec{a}_2\} = \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.