Math 54, midterm 1

Instructor: Semyon Dyatlov

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Name: ______________________________ SID: ________________

Problem 1: ________ / 10

Problem 2: ________ / 10

Problem 3: ________ / 10

Problem 4: ________ / 10

Problem 5: ________ / 10

Total: ________ / 50

- Write your solutions in the space provided. Do not use your own paper. I can give you extra paper if needed. Indicate clearly where your answer is.
- Explain your solutions as clearly as possible. This will help me find what you did right and what you did wrong, and award partial credit if possible.
- Justify all your steps. (Problem 3 is exempt from this rule.) A correct answer with no justification will be given 0 points. Pictures without explanations are not counted as justification. You may cite a theorem from the book by stating what it says.
- No calculators or notes are allowed on the exam, except for a single two-sided 5"×9" sheet of hand-written notes. Cheating will result in academic and/or disciplinary action. Please turn off cellphones and other electronic devices.

1. (a) Find the standard matrix of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula $T(x_1, x_2, x_3) = (x_1 + 2x_2, x_2 + 2x_3)$.

(b) Describe the solution set of the equation $T(\vec{x}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in parametric vector form.

2. (a) Let A be a 3×3 matrix such that $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$. Solve the equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) Find a linear dependence relation between the vectors

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \ \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{a}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

3. Circle the correct answer for each of the following true/false or multiple choice questions, or write in the correct answer in the blank space provided. No justification is required. For each question, you get 1 point if you answer it correctly, 0 points if you choose not to answer, and -1 point if you answer it incorrectly. (If your total score for the problem is negative, it will be changed to zero.)

(a) For each matrix A, $\det(A^T A) \geq 0$.

True False

(b) For all square matrices A and B of the same size, if $AB = 0$, then $A = 0$ or $B = 0$.

True False

(c) For all square matrices A and B of the same size, $\det(A^2 - B^2) = \det(A + B) \cdot \det(A - B)$.

True False

(d) If the equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$, then A is square and $\det A \neq 0$.

True False

(e) If AB is well-defined and invertible and A is a square matrix, then B is invertible.

True False

(f) If A has size $m \times n$ and B has size ____ $\times$ ____, then AB is well-defined and has size ____ $\times k$.

(g) If a linear transformation $T : \mathbb{R}^a \to \mathbb{R}^b$ is onto, then:

$a \leq b$ $a \geq b$

(h) If the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^4$ form a linearly dependent set, then each of these vectors is a linear combination of the other two.

True False

(i) For each matrix A, if the equation $A\vec{x} = \vec{0}$ has unique solution, then for each $\vec{b}$, the number of solutions to the equation $A\vec{x} = \vec{b}$ is either 0 or 1.

True False

(j) If A is a square matrix and $A^T A = I$, then $AA^T = I$.

True False

4. (a) Find all $t \in \mathbb{R}$ such that the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + t \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ has linearly dependent columns.

(b) Find $\det \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 5 \\ 1 & 3 & 0 & 5 \end{bmatrix}$.

5. Solve **one** of the following two problems. Mark which one you want graded.

(a) Assume that A and B are square matrices of the same size and $A^2 = B^2 = (AB)^2 = I$. Show that $AB = BA$.

(b) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^n$. Prove that $\mathrm{Span}\{\vec{a}_1, \vec{a}_2\} \subset \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Here the sign '$\subset$' means 'is contained in'. Also, prove that if $\vec{a}_3 \in \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$, then $\mathrm{Span}\{\vec{a}_1, \vec{a}_2\} = \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.