MATH 54-1, H/W 9

5.3 (12) Yes; $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(24) No, by Th. 7 (b).

(28) A has n LIN. IND. eigenvectors $\to$ A is diagonalizable. Write $A = PDP^{-1}$ with D diagonal; then $A^T = (P^{-1})^T D^T P^T$; however, $(P^{-1})^T = (P^T)^{-1}$ and $D^T = D$. So, $A^T = QDQ^{-1}$ with $Q = (P^T)^{-1}$ invertible. Therefore, A^T is diagonalizable & it has n LIN. IND. eigenvectors.

5.4 (2) $\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$.

(6) (a) $T(2-t+t^2) = 2-t+3t^2-t^3+t^4$.

(b) For any polynomials $f, g \in \mathbb{P}_2$ and any $c, d \in \mathbb{R}$,

$T(cf+dg) = (cf+dg)\cdot(1+t^2) = c(1+t^2)f + d(1+t^2)g = cT(f) + dT(g)$.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(9) (a) $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$ (b) For any $f, g \in \mathbb{P}^2$ and $c, d \in \mathbb{R}$, we have $T(cf+dg) = \begin{bmatrix} (cf+dg)(-1) \\ (cf+dg)(0) \\ (cf+dg)(1) \end{bmatrix} = \begin{bmatrix} cf(-1)+dg(-1) \\ cf(0)+dg(0) \\ cf(1)+dg(1) \end{bmatrix} = c\begin{bmatrix} f(-1) \\ f(0) \\ f(1) \end{bmatrix} + d\begin{bmatrix} g(-1) \\ g(0) \\ g(1) \end{bmatrix} = cT(f) + dT(g)$.

(c) We have $T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

$T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\to$ $[T] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(12) Following Example 4, if $P = [\vec{b}_1 \; \vec{b}_2] = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, then

$[T]_{\mathcal{B}} = P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

(14) The eigenvalues are 8, with eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and -2, with eigenv. $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$. So, if $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$, then $[T]_{\mathcal{B}} = \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix}$.

(20) A similar to B $\to \exists P: A = PBP^{-1} \to A^2 = PBP^{-1} \cdot PBP^{-1} = PB^2P^{-1} \to$ A^2 similar to B^2.

(22) A diagonalizable $\rightarrow$ $A = PDP^{-1}$, D diagonal.

B similar to A $\rightarrow$ $B = QAQ^{-1} = Q(PDP^{-1})Q^{-1} =$

$= (QP)D(QP)^{-1} \Rightarrow$ B is diagonalizable.

4.7 (5) (a) $P_{\mathcal{B}\leftarrow\mathcal{A}} = [[\vec{a}_1]_{\mathcal{B}} \; [\vec{a}_2]_{\mathcal{B}} \; [\vec{a}_3]_{\mathcal{B}}] = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

(b) $[\vec{x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}}[\vec{x}]_{\mathcal{A}} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$.

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(14) Let $\mathcal{C} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Then

$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[1-3t^2]_{\mathcal{C}} \; [2+t-5t^2]_{\mathcal{C}} \; [1+2t]_{\mathcal{C}}] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$.

If $f = t^2$, then $[f]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P_{\mathcal{C}\leftarrow\mathcal{B}}[f]_{\mathcal{B}}$. Solving this equation: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}[f]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we get $[f]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.