MATH 54-1, H/W 7

4.4 (11) Write $P_{\mathcal{B}} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$; then

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

(14) We need to find c_1, c_2, c_3:

$$3 + t - 6t^2 = c_1(1-t^2) + c_2(t-t^2) + c_3(2-2t+t^2).$$

This is the same as saying

$$3 + t - 6t^2 = (c_1 + 2c_3) + (c_2 - 2c_3)t + (-c_1 - c_2 + c_3)t^2;$$

we get the SLE

$$\begin{cases} c_1 + 2c_3 = 3 \\ c_2 - 2c_3 = 1 \\ -c_1 - c_2 + c_3 = -6 \end{cases}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$$

(18) For each k, $\vec{b}_k = 0 \cdot \vec{b}_1 + \cdots + 1 \cdot \vec{b}_k + \cdots + 0 \cdot \vec{b}_n$, so $[\vec{b}_k]_{\mathcal{B}} = (0, \ldots, 1, \ldots, 0) = \vec{e}_k$.

(28) Translate everything into coordinate language (w.r.t. the basis $\{1, t, t^2, t^3\}$):

$1 - 2t^2 - 3t^3 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ -2 \\ -3 \end{bmatrix}$, $t + t^3 \rightsquigarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $1 + 3t - 2t^2 \rightsquigarrow \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$

We need to find out whether these coordinate vectors are linearly independent. The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix}$ does not have a pivot in every column; thus, the original polynomials are linearly dependent.

4.5 (9) $\left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} = \mathrm{Col} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$; the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ has 2 pivot columns, so the dimension is 2.

(10) $H = \mathrm{Col}\, A$, where $A = \begin{bmatrix} 2 & -4 & -3 \\ -5 & 10 & 6 \end{bmatrix} \xrightarrow[\text{RED}]{\text{Row}} \begin{bmatrix} \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$

The dimension is 2. (In fact, $H = \mathbb{R}^2$.)

(14) A has 3 pivot columns $\to$ dim Col $A = 3$.

The ~~sys~~ equation $A\vec{x} = \vec{0}$ has 3 free variables $\to$ dim Nul $A = 3$.

(26) ~~Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be~~ Since dim H, it has a basis $\mathcal{B}$ of n vectors. The vectors in $\mathcal{B}$ are lin. ind.; since dim $V = n$, by the Basis Theorem, $\mathcal{B}$ is a basis for V. Then, $V = \mathrm{Span}\, \mathcal{B} = H$.

4.6 (6) dim Row A = dim Col A = rank $A = 3$ = rank A^T.

dim Nul $A = 3 -$ rank $A = 0$.

(10) $6 =$ dim Nul A + dim Col A $\rightarrow$ dim Col $A = 1$.

(14) dim Row A = rank A = number of pivot positions in A.
If ~~a~~ A is either 3×4 or 4×3, the maximal number of its pivot positions is 3.

(15) dim Nul $A = 8 -$ rank A; the maximal ~~of~~ rank A is 6, so the minimal dim Nul A is 2.

(20) We get $A\vec{x}=\vec{b}$ consistent ~~with 2D~~ space of ~~solutions~~, when A is 6×8. ~~We get by the characterization~~
Also, A ~~has~~ 2 free variables $\rightarrow$ A has 6 pivots $\rightarrow$ A has a pivot in each row $\rightarrow$ $A\vec{x}=\vec{b}$ is consistent for all $\vec{b}$. So, it is impossible.