MATH 54-1, H/w 14

10.2 (3) $y''+4y=0 \to y = c_1\cos(2x)+c_2\sin(2x)$

$y' = -2c_1\sin(2x)+2c_2\cos(2x)$

$0 = y(0) = c_1$

$0 = y'(\pi) = 2c_2 \to c_1=c_2=0$, no nonzero solutions

(11) $\begin{cases} y''+\lambda y=0 \\ y(0)=y(2\pi) \\ y'(0)=y'(2\pi)\end{cases} \to$ Cases:

$\underline{\lambda=0} \to y = c_1+c_2x \to$

$\to c_1 = y(0) = \overset{y(2\pi)=}{c_1+2\pi c_2}$

$c_2 = y'(0)=y'(2\pi) = c_2$

$\downarrow$

need only $c_2=0 \to$

$\to$ eigenvalue $\lambda=0$, with eigenfunction $y(x)\equiv 1$.

$\underline{\lambda<0} \to$ if we had a nonzero solution $y(x)$, then $0 = \int_0^{2\pi}(y''+\lambda y)\cdot y\,dx =$

$= y(x)y'(x)\Big|_{x=0}^{2\pi} - \int_0^{2\pi} y'(x)^2-\lambda y(x)^2dx.$

But $y(0)y'(0) = y(2\pi)y'(2\pi)$ by boundary conditions $\to$

$\to \int_0^{2\pi} y'(x)^2-\lambda y(x)^2\,dx = 0.$

The expr. under the integral is ≥ 0, as $\lambda<0 \to y'(x)^2-\lambda y(x)^2=0 \to$

$\to \lambda y(x)^2=0 \to y\equiv 0$, a contradiction.

No eigenvalues $\lambda<0$

$\underline{\lambda>0} \to y = c_1\cos(\sqrt{\lambda}\,x)+c_2\sin(\sqrt{\lambda}\,x)$

$c_1 = y(0) = y(2\pi) = c_1\cos(2\pi\sqrt{\lambda})+c_2\sin(2\pi\sqrt{\lambda})$

$\sqrt{\lambda}c_2 = y'(0) = y'(2\pi) = -c_1\sqrt{\lambda}\sin(2\pi\sqrt{\lambda})+c_2\sqrt{\lambda}\cos(2\pi\sqrt{\lambda})$

Two equations on c_1, c_2:

$c_1(\cos(2\pi\sqrt{\lambda})-1)+c_2\sin(2\pi\sqrt{\lambda})=0$

$-c_1\sin(2\pi\sqrt{\lambda}) + c_2(\cos(2\pi\sqrt{\lambda})-1)=0$

To have a nonzero solution, we need

$0 = \det\begin{bmatrix}\cos(2\pi\sqrt{\lambda})-1 & \sin(2\pi\sqrt{\lambda}) \\ -\sin(2\pi\sqrt{\lambda}) & \cos(2\pi\sqrt{\lambda})-1\end{bmatrix} = \cos^2(2\pi\sqrt{\lambda})-2\cos(2\pi\sqrt{\lambda})+1+\sin^2(2\pi\sqrt{\lambda}) = 2(1-\cos(2\pi\sqrt{\lambda})) \Leftrightarrow$

$\Leftrightarrow \cos(2\pi\sqrt{\lambda})=1 \Leftrightarrow 2\pi\sqrt{\lambda} = 2\pi k,\ k\in\mathbb{Z},\ k>0 \Leftrightarrow \lambda=k^2.$

For $\lambda=k^2$, we get $\cos(2\pi\sqrt{\lambda})-1=0$, $\sin(2\pi\sqrt{\lambda})=0$,

$\to c_1, c_2$ are both free. Thus, we get eigenvalues k^2, $k\geq 0$, $k\in\mathbb{Z}$, w/ eigenfunctions $\cos(kx)$, $\sin(kx)$

(i.e., basis of the corresponding eigenspace is $\{\cos(kx), \sin(kx)\}$)

(13) $\underline{\lambda \le 0}$ — Similarly to the previous problem,

$$0 = -\int_0^{\pi} (y''+\lambda y)y\,dx = y(0)y'(0) - y(\pi)y'(\pi) + \int_0^{\pi} y'(x)^2 - \lambda y(x)^2\,dx.$$

Using boundary conditions, we get $0 = y(0)^2 + \int_0^{\pi} y'(x)^2 - \lambda y(x)^2\,dx$.

Since $\lambda \le 0$, all terms are nonnegative $\to$ $y(0)=0$, $y' \equiv 0$, $\lambda y \equiv 0$. Since $y' \equiv 0$, $y = \text{const}$; but $y(0)=0 \Rightarrow y \equiv 0$, a contradiction.

So, no eigenvalues with $\lambda \le 0$.

$\underline{\lambda > 0}$ $\to$ $y = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

$y' = -\sqrt{\lambda}C_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}C_2 \cos(\sqrt{\lambda}x)$

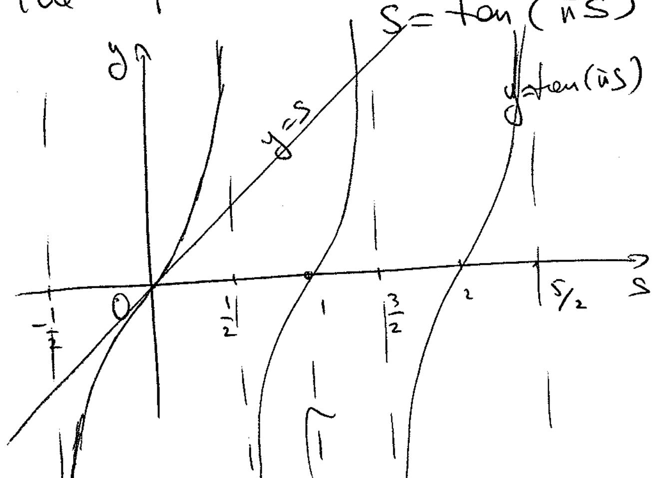
$$\cancel{0 = y^*(0) - y'(0) = C_1(\cos(\sqrt{\lambda}x) + \sqrt{\lambda}\sin(\sqrt{\lambda}x)) + C_2(\sin(\sqrt{\lambda}x) - \sqrt{\lambda}\cos(\sqrt{\lambda}x))}$$
$$\cancel{0 = y^*(\pi) = C_1 \cos ((}$$

$$(*)\begin{cases} 0 = y(0) - y'(0) = C_1 - C_2\cdot\sqrt{\lambda} \\ 0 = y(\pi) = C_1\cos(\pi\sqrt{\lambda}) + C_2\sin(\pi\sqrt{\lambda}) \end{cases}$$

Need $0 = \begin{vmatrix} 1 & -\sqrt{\lambda} \\ \cos(\pi\sqrt{\lambda}) & \sin(\pi\sqrt{\lambda}) \end{vmatrix} = \sin(\pi\sqrt{\lambda}) - \sqrt{\lambda}\cos(\pi\sqrt{\lambda}) \iff$

$\iff \sqrt{\lambda} = \tan(\pi\sqrt{\lambda})$ [$\cos \ne 0$, as else $\sin = 0$, a contradiction]

The equation ~~$\sqrt{\lambda} = \tan(\pi\sqrt{\lambda})$~~ $s = \tan(\pi s)$ has a root s_n in each interval $[n-\frac{1}{2}, n+\frac{1}{2}]$ $\to$



$\to$ the corresponding eigenvalue is $\lambda_n = s_n^2$; a solution of the system (*) is then given by a solution of its first equation (as the second equation has to be a multiple of the first when det $=0$)

$\leftarrow$ take $C_1 = \sqrt{\lambda}$, $C_2 = 1$

Eigenfunction $\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x)$, $n \ge 1$, $n \in \mathbb{Z}$

10.3 (12) $a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\cdot\frac{\pi^3}{3} = \frac{\pi^2}{3}$

If $n>0 \to$ $a_n = \frac{1}{\pi}\int_0^{\pi} x^2 \cos(nx)\,dx =$

$$= \frac{1}{n\pi}\int_0^{\pi} x^2\, d(\sin(nx)) =$$

$$= \underbrace{\frac{1}{n\pi}x^2\sin(nx)\Big|_{x=0}^{\pi}}_{0} - \frac{2}{n\pi}\int_0^{\pi} x\sin(nx)\,dx =$$

$$= \frac{2}{n^2\pi}\int_0^{\pi} x\, d(\cos(nx))$$

$(\cos(n\pi) = (-1)^n)$

$$= \frac{2}{\pi n^2}x\cos(nx)\Big|_{x=0}^{\pi} - \frac{2}{\pi n^2}\int_0^{\pi}\cos(nx)\,dx =$$

$$= \frac{2(-1)^n}{n^2} - \frac{2}{\pi n^3}\sin(nx)\Big|_{x=0}^{\pi} = \frac{2(-1)^n}{n^2}.$$

$n>0 \longrightarrow$ $b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx = -\frac{1}{n\pi}\int_0^{\pi} x^2\, d(\cos(nx)) =$

$$= -\frac{1}{n\pi}x^2\cos(nx)\Big|_{x=0}^{\pi} + \frac{2}{n\pi}\int_0^{\pi} x\cos(nx)\,dx =$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{n^2\pi}\int_0^{\pi} x\, d(\sin(nx)) =$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{\pi n^2}x\sin(nx)\Big|_{x=0}^{\pi} - \frac{2}{\pi n^2}\int_0^{\pi}\sin(nx)\,dx =$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{\pi n^3}\cos(nx)\Big|_{x=0}^{\pi} = -\frac{\pi}{n}(-1)^n + \frac{2}{\pi n^3}[(-1)^n - 1]$$

So, $f(x) \sim \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}\cos(nx) + \frac{(-1)^n(2-\pi^2 n^2) - 2}{\pi n^3}\sin(nx)$

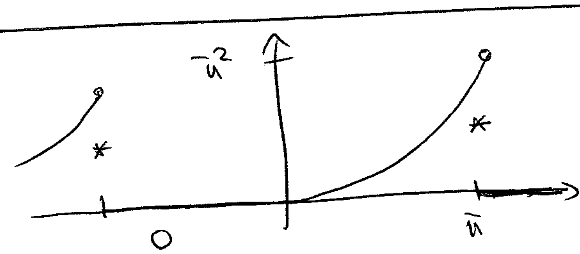
---

(20) The series converges to:

0, for $-\pi < x \le 0$

x^2, for $0 < x < \pi$

$\pi^2/2$, for $x = \pm\pi$.



---

(28) (a) Long and boring calculation, very similar to the one in problem 12.
(The sin coefficients are zero since our function is even.)

(b) Take $x=0$ and apply the Fourier series convergence theorem:

$$0 = 0^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

(c) We found $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. But $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} =$

$$= \cancel{\sum_{\substack{n=2m\\ m\geq 1}} \frac{-1}{n^2} - \sum_{\substack{n=2m+1\\ m\geq 1}}} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots =$$

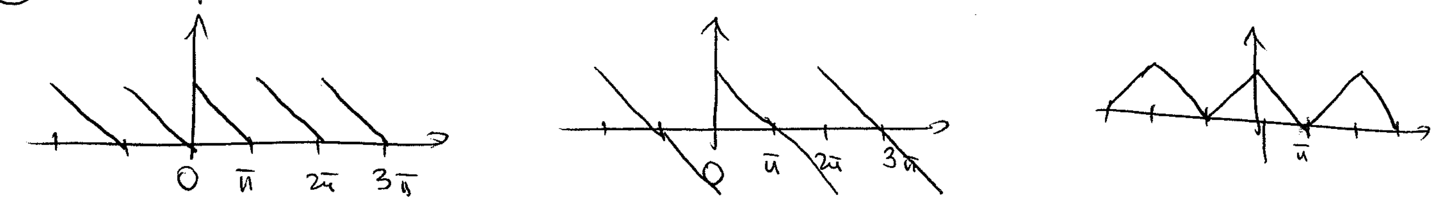
$$= \frac{1}{1^2} + \frac{1}{2^2} - \frac{2}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{2}{4^2} + \cdots =$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right) =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since this sum is $\frac{\pi^2}{12}$, we get $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

10.4 (4) π-periodic odd 2π-periodic even 2π-periodic



---

(6) $b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \cdot \sin(nx)\,dx = \frac{1}{\pi} \int_0^{\pi} \sin((n+1)x) + \sin((n-1)x)\,dx =$

$\boxed{\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))}$

$$= -\frac{1}{\pi} \left\{ \left[\frac{1}{n+1} \cos(n+1)x \Big|_{x=0}^{\pi} + \frac{1}{n-1} \cos(n-1)x \Big|_{x=0}^{\pi} \right] \right. =$$

$$= -\frac{1}{\pi} \cdot \left[(-1)^{n+1} - 1\right] \cdot \left[\frac{1}{n+1} + \frac{1}{n-1}\right] =$$

$$= \frac{[1 - (-1)^{n+1}] \cdot 2n}{\pi(n^2-1)} = \begin{cases} 0, & \text{if } n \text{ is } \cancel{\text{even}} \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & \text{if } n \text{ is } \cancel{\text{odd}} \text{ even} \end{cases}$$

So, $f(x) \sim \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{k}{\pi(4k^2-1)} \sin(2kx)$.

10.5

(2) Using problem 10.4.7, we find

$$x^2 \sim \sum_{n=1}^{\infty} \left[\frac{2\pi(-1)^{n+1}}{n} + \frac{4}{\pi n^3}((-1)^n-1) \right] \sin(nx)$$

(We use $\sin(nx)$ because of the boundary conditions)

So, $$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2\pi(-1)^{n+1}}{n} + \frac{4}{\pi n^3}((-1)^n-1) \right] e^{-n^2 t} \sin(nx)$$

is our solution.

---

(7) Get rid of the boundary conditions: $u(x,t) = 5 + \frac{5x}{\pi} + w(x,t)$,

where $$\begin{cases} \dfrac{\partial w}{\partial t} = 2 \dfrac{\partial^2 w}{\partial x^2} \\ w(0,t) = 0 = w(\pi,t) \\ w(x,0) = \sin(3x) - \sin(5x) - 5 - \dfrac{5x}{\pi}. \end{cases}$$

---

(8) Get rid of the boundary conditions: $u(x,t) = 3x + w(x,t)$,

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} \\ w(0,t) = 0 = w(\pi,t) \\ w(x,0) = -3x \end{cases} \longrightarrow \quad w(x,t) = 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin(nx)$$

$$u(x,t) = 3x + 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin(nx).$$