Math 54, final exam information and review

August 10, 2010

1 General information

The final exam will take place on Friday, August 13, from 8–10 AM in room 2 Evans. The exam itself will start at 8:10, but I ask you to come at 8 so that I could hand out the exams and everybody would start at the same time. There are no calculators and no materials allowed, except for one two-sided A4 sheet of hand-written notes. Do not bring your own paper — I will provide extra sheets if needed. The final exam will cover the entire course, with an emphasis on the material not covered by the two midterms.

2 Sample computational problems

Note: the final exam may contain computational problems on material covered by the midterms; samples of these are not included here.

1. Given the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},\ \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

(a) Use Gram–Schmidt to find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by these three vectors.

(b) Find an orthonormal basis for V.

(c) Use the orthogonal basis you found in (a) to find the orthogonal projection of the vector $\vec{b} = (1, 2, 3, 4)$ onto V.

(d) Let A be the matrix whose columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and let $\vec{b}$ be the vector from (c). Find the least-squares solution to the equation $A\vec{x} = \vec{b}$. Find the least-squares error.

2. Consider the following inner product on $\mathbb{P}_2$:

$$\langle f, g\rangle = f(0)g(0) + \int_0^1 f(t)g(t)\,dt,\ \ f, g \in \mathbb{P}_2,$$

and the subspace

$$V = \{f \in \mathbb{P}_2 \mid f(1) = 0\}.$$

(a) Find a basis for V. To do this, you can write the equations that the coefficients of an element of V (in other words, its coordinates in the standard basis $\{1, t, t^2\}$) have to satisfy, and find a basis for the space of coefficients of elements of V by representing it as $\operatorname{Nul} A$ for some matrix A.

(b) Use Gram–Schmidt and the result of (a) to find an orthogonal basis of V.

3. Find the solution to the initial value problem

$$y''(x) + 2y'(x) + 2y(x) = e^{-x}(x + 2\cos x - 3\cos(2x)),\ y(0) = 1,\ y'(0) = 2.$$

4. (a) Find all possible values of $\lambda \in \mathbb{R}$ for which the problem

$$y''(x) + \lambda y(x) = 0,\ 0 < x < 1;$$
$$y(0) = 0,\ y'(1) = 0$$

has a nonzero solution; for each of these λ, find a basis of the set of solutions to the problem above.

(b) Find the basic solutions of the following problem that can be obtained using separation of variables (i.e., have the form $X(x)T(t)$):

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1,\ t > 0;$$
$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(1,t) = 0,\ t > 0.$$

5. (a) Find the Fourier sine and cosine series of the function $f(x) = 1+x,\ 0 < x < \pi$. Find the functions to which these series converge and sketch their graphs.

(b) Find the formal solution of the following problem for the heat equation (note the inhomogeneous boundary conditions):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\ t > 0;$$
$$u(0,t) = 1,\ u(\pi,t) = 1 + \pi,\ t > 0;$$
$$u(x,0) = 0,\ 0 < x < \pi.$$

Find the (pointwise in x) limit of this solution as $t \to +\infty$.

(c) Find the formal solution of the following problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\ t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0,\ t > 0;$$
$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 1 + x,\ 0 < x < \pi.$$

6. Find the full Fourier series of the function $f(x) = e^x$, $x \in [-\pi, \pi]$. (Hint: to compute the integrals, integrate by parts to get a system of linear equations on the coefficients a_k and b_k.) Determine the function to which this Fourier series converges.

7. Find a fundamental system and the general solution of the equation

$$y''' - y = 0.$$

Find the solution satisfying the initial conditions

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 1.$$

8. Consider the problem

$$\frac{\partial u}{\partial t}(x,t) = (x+1)\frac{\partial u}{\partial x}(x,t),\ x \in \mathbb{R}, t > 0;$$
$$u(x,0) = x - 1,\ x \in \mathbb{R}.$$

We will look for a solution in the form

$$u(x,t) = a_0(t) + a_1(t)x,$$

where a_0 and a_1 are some functions.

(a) Verify that in order for $u(x,t)$ to be a solution to the problem above, the functions a_0 and a_1 have to solve the initial value problem

$$a_0'(t) = a_1(t),\ a_1'(t) = a_1(t),$$
$$a_0(0) = -1,\ a_1(0) = 1.$$

(Hint: freeze t and regard both sides of the PDE as polynomials in x; make their coefficients equal.)

(b) Write the system above in normal form and find the solution of this IVP for a system of ODE; use it to write a solution to the original IVP for the PDE.

9. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in some inner product space, and assume that

$$\|\vec{u}\| = 1,\ \|\vec{v}\| = 2,$$
$$\langle \vec{u}, \vec{v} \rangle = 0,\ \langle \vec{u}, \vec{w} \rangle = 3,\ \langle \vec{v}, \vec{w} \rangle = 4.$$

What is the minimal possible value of $\|\vec{w}\|$? (Hint: use the orthogonal projection formula and Pythagorean theorem.)

3 Sample theoretical problems

1. Using Cauchy–Schwarz inequality, prove that for every continuous function f on the interval $[0, 5]$,

$$\int_0^5 f(x)\,dx \le \sqrt{5}\left(\int_0^5 f(x)^2\,dx\right)^{1/2}.$$

2. For each of the following pairs of subspaces V, W of the space of continuous functions on $[-1, 1]$ with the inner product given by the integral of the product of the two functions over $[-1, 1]$, prove that each element of V is orthogonal to each element of W:

(a) V is the space of all odd functions and W is the space of all even functions

(b) V is the space of all functions f such that $f(x) = 0$ for each $x \in [0, 1]$; W is the space of all functions g such that $g(x) = 0$ for each $x \in [-1, 0]$.

3. Assume that $y(x)$ is a (twice continuously differentiable) function on $\mathbb{R}$ such that

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where p and q are continuous functions on $\mathbb{R}$. Let $f(x) = y(x)^2 + (y'(x))^2$. Using the existence/uniqueness theorem for ODE, prove that either the function f is identically zero or the equation $f(x) = 0$ has no solutions.

4. Assume that A is a 2×2 matrix such that $A(3, -2) = (-3, 2)$ and the rank of A is equal to 1. Is A diagonalizable?

5. Assume that A is an invertible 3×3 matrix with **integer** elements. Prove that A^{-1} has integer elements if and only if $|\det A| = 1$. (Hint: recall the cofactor expansions of the determinant, multiplicativity of determinants, and the formula for the inverse in Lay, Section 3.3.)

6. Assume that u and v are two solutions to the differential equation $u'' + p(x)u' + q(x)u = 0$. Let $W = uv' - u'v$ be the Wronskian of u and v. Find a (nontrivial) first order linear differential equation satisfied by W. The coefficients of this equation can contain the functions p and q, but not u or v. (Hint: differentiate W once, and use the equations satisfied by u and v to replace u'' and v'' with some expressions featuring only u, v, u', v', p, q.)

7. Let V be the space of all solutions to the equation $y''' - y = 0$.

(a) Prove that if $y \in V$, then $y' \in V$. (Hint: differentiate the differential equation.)

(b) Let $T : V \to V$ be the linear transformation defined by the formula $T(y) = y'$. Prove that T^3 is the identity transformation.

4 Answers to computational problems

1. (a) $\{(1,1,0,0),(-1/2,1/2,1,0),(1/3,-1/3,1/3,1)\}$
(b) $\{(1,1,0,0)/\sqrt{2},(-1,1,2,0)/\sqrt{6},(1,-1,1,3)/2\sqrt{3}\}$
(c) $(3/2,3/2,7/2,7/2)$.
(d) The normal system is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix};$$

the least-squares solution is $(3/2,0,7/2)$ and the least-squares error is 1.

2. (a) A polynomial $a_0 + a_1 t + a_2 t^2$ lies in V if and only if the coefficient vector $\vec{a} = (a_0, a_1, a_2)$ solves the equation

$$a_0 + a_1 + a_2 = 0.$$

Thus, $\vec{a}$ lies in $\operatorname{Nul} A$ for $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$; a basis for this space is $\{(1,-1,0),(1,0,-1)\}$. Therefore, a basis for V is $\{1-t, 1-t^2\}$.

(b) We find $\|1-t\|^2 = 4/3$, $\langle 1-t, 1-t^2 \rangle = 17/12$; the orthogonal basis is $\{1-t, -\frac{1}{16} + \frac{17}{16}t - t^2\}$.

3. The trial solution is

$$y(x) = (A_0 + A_1 x)e^{-x} + xe^{-x}(B_1 \cos x + B_2 \sin x) + e^{-x}(C_1 \cos(2x) + C_2 \sin(2x));$$

we get

$$y'' + 2y' + 2y = (A_0 + A_1 x)e^{-x} + 2xe^{-x}(B_2 \cos x - B_1 \sin x) \\ -3e^{-x}(C_1 \cos(2x) + C_2 \sin(2x));$$

therefore,

$$A_0 = 0,\ A_1 = 1,\ B_1 = 0,\ B_2 = 1,\ C_1 = 1,\ C_2 = 0;$$

the general solution to the inhomogeneous equation is

$$y = e^{-x}(x + x\sin x + \cos(2x) + c_1 \cos x + c_2 \sin x);$$

the initial conditions give

$$1 = y(0) = 1 + c_1,\ 2 = y'(0) = c_2 - c_1;$$

therefore, the solution is

$$y = e^{-x}(x + x\sin x + \cos(2x) + 2\sin x).$$

4. (a) $\lambda = (\pi(k+1/2))^2$, where $k \in \mathbb{Z}$, $k \geq 0$. The corresponding eigenfunction is $\sin(\pi(k+1/2)x)$.

(b) The basic solutions are

$$u_k(x,t) = e^{-t/2}\cos(\nu_k t)\sin(\pi(k+1/2)x),$$
$$v_k(x,t) = e^{-t/2}\sin(\nu_k t)\sin(\pi(k+1/2)x),$$

where

$$\nu_k = \sqrt{\pi^2\Big(k+\frac{1}{2}\Big)^2 - \frac{1}{4}};\ k\in\mathbb{Z},\ k\geq 0.$$

5. (a)

$$1+x \sim \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\sin((2j-1)x)}{2j-1} + 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin(kx)}{k};$$
$$1+x \sim 1+\frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos((2j-1)x)}{(2j-1)^2}.$$

(b)

$$u(x,t) = 1+x-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{e^{-(2j-1)^2 t}\sin((2j-1)x)}{2j-1} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}e^{-k^2 t}\sin(kx)}{k}.$$

We have $\lim_{t\to+\infty} u(x,t) = 1+x$.

(c)

$$u(x,t) = \Big(1+\frac{\pi}{2}\Big)t - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\sin((2j-1)t)\cos((2j-1)x)}{(2j-1)^3}.$$

6. We have $a_0 = (e^{\pi}-e^{-\pi})/\pi$. Now, fix $k>0$; integration by parts gives

$$a_k = -\frac{b_k}{k},\ b_k = \frac{(-1)^{k+1}}{\pi k}(e^{\pi}-e^{-\pi}) + \frac{a_k}{k}.$$

Solving this as a system of linear equations on a_k and b_k, we find

$$a_k = \frac{(-1)^k(e^{\pi}-e^{-\pi})}{\pi(k^2+1)},\ b_k = \frac{(-1)^{k+1}(e^{\pi}-e^{-\pi})k}{\pi(k^2+1)};$$

therefore,

$$e^x \sim \frac{e^{\pi}-e^{-\pi}}{\pi}\left(\frac{1}{2}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2+1}(\cos(kx)-k\sin(kx))\right).$$

The Fourier series converges to the 2π-periodic continuation of the function equal to e^x on $(-\pi,\pi)$ and to $(e^{\pi}+e^{-\pi})/2$ at $\pm\pi$.

7. The fundamental system is $\{e^x, e^{-x/2}\cos(\sqrt{3}x/2), e^{-x/2}\sin(\sqrt{3}x/2)\}$; the general solution is $c_1e^x + c_2e^{-x/2}\cos(\sqrt{3}x/2) + c_3e^{-x/2}\sin(\sqrt{3}x/2)$; the solution with the given initial conditions is $\frac{2}{3}e^x - \frac{2}{3}e^{-x/2}\cos(\sqrt{3}x/2)$.

8. (a) We have for $u = a_0(t) + a_1(t)x$,

$$\frac{\partial u}{\partial t} = a_0'(t) + a_1'(t)x, \; (x+1)\frac{\partial u}{\partial x} = a_1(t) + a_1(t)x.$$

(b) For $\vec{a}(t) = (a_0(t), a_1(t))$, $\vec{a}'(t) = A\vec{a}(t)$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The general solution is $c_1(1,0) + c_2e^t(1,1)$; the solution to the IVP is $-2(1,0) + e^t(1,1)$. The corresponding solution of the PDE is $u(x,t) = e^t - 2 + e^t x$.

9. The orthogonal projection of $\vec{w}$ onto the space spanned by $\vec{u}$ and $\vec{v}$ is $3\vec{u} + \vec{v}$. Then $\vec{w} = 3\vec{u} + \vec{v} + \vec{a}$, where $\vec{a}$ is an arbitrary vector orthogonal to $\vec{u}$ and $\vec{v}$. The length of $\vec{w}$ is minimized when $\vec{a} = 0$ and in this case it is equal to $\sqrt{7}$.

5 Hints and answers for theoretical problems

1. Use the space of continuous functions on $[0,5]$ with the inner product

$$\langle f, g \rangle = \int_0^5 f(x)g(x)\,dx$$

and apply Cauchy–Schwarz to the functions f and 1.

2. (a) If $f \in V$ and $g \in W$, then the product $f(x)g(x)$ is an odd function (b) If $f \in V$ and $g \in W$, then $f(x)g(x) = 0$ for all x.

3. Assume that the equation $f(x) = 0$ has a solution $x = x_0$; we will prove that f is identically zero. Since $f(x_0) = 0$, the function y solves the IVP

$$y'' + p(x)y' + q(x)y = 0, \; y(x_0) = y'(x_0) = 0.$$

On the other hand, the zero function also solves this IVP. By the uniqueness part of the existence/uniqueness theorem for linear ODE, $y \equiv 0$; therefore, $f \equiv 0$.

4. The vector $(3,-2)$ is an eigenvector of A with eigenvalue -1. Since the rank of A is equal to 1, it is not invertible; this, 0 is an eigenvalue of A. Since A has two distinct eigenvalues and it is a 2×2 matrix, it is diagonalizable.

5. The cofactor expansions show that if A is a matrix with integer entries, then $\det A$ is an integer number. If A^{-1} has integer entries, then $1 = \det A \cdot \det A^{-1}$, where both factors on the right-hand side are integer; thus, $\det A = \pm 1$. On the other hand, if $\det A = \pm 1$, then $1/\det A$ is integer; the adjugate of A is integer because it consists of ± 1 times determinants of matrices of integers; by the formula for the inverse matrix, A^{-1} has integer entries.

6. We have $W' = uv'' - u''v = u(-p(x)v' - q(x)v) - v(-p(x)u' - q(x)u) = -p(x)W$; therefore, W solves the equation $W'(x) + p(x)W(x) = 0$.