Consider the equation

$$y'' - y = 5e^{-2t} + \cos t = f(t).$$

(1.) Let V be the set of all functions of the form $y = Ae^{-2t} + B\cos t + C\sin t$, $A, B, C \in \mathbb{R}$. Then it is a subspace of the space of all functions. Indeed,

- $A = B = C = 0 \Rightarrow y = 0 \Rightarrow 0 \in V$
- If $y_1, y_2 \in V$, then $y_1 = A_1 e^{-2t} + B_1\cos t + C_1 \sin t$, $y_2 = A_2 e^{-2t} + B_2 \cos t + C_2 \sin t$

for some $A_j, B_j, C_j \in \mathbb{R}$. Then

$$y_1 + y_2 = (A_1 + A_2)e^{-2t} + (B_1 + B_2)\cos t + (C_1 + C_2)\sin t \in V.$$

- If $y \in V$ and $c \in \mathbb{R}$, then

$$y = Ae^{-2t} + B\cos t + C\sin t \quad \text{for some } A, B, C \in \mathbb{R}$$

Then, $c \cdot y = (cA)e^{-2t} + (cB)\cos t + (cC)\sin t \in V$.

Another proof: the map $\mathbb{R}^3 \to V$ given by $\begin{bmatrix} A \\ B \\ C \end{bmatrix} \mapsto Ae^{-2t} + B\cos t + C\sin t$ is linear and V is its image; therefore, V is a subspace.

Dimension: $V = \operatorname{span}\{e^{-2t}, \cos t, \sin t\}$ (yet another proof that V is a subspace). ~~So~~ But these 3 functions are linearly independent. Indeed, assume that $y = Ae^{-2t} + B\cos t + C\sin t = 0$. Then, if $A \neq 0$, then $\lim_{t\to-\infty} y(t) = \pm\infty$. So, $A = 0$. Now, $y(0) = 0 = B$, $y(\pi/2) = 0 = C$.

So, $\dim V = 3$.

(2.) The operator $T: V \to$ (space of all fns) given by $T(y) = y'' - y$ is linear (because differentiation is linear). Moreover, its image lies in V. Indeed, if

$$y = Ae^{-2t} + B\cos t + C\sin t, \text{ then}$$

$$y'' = 4Ae^{-2t} - B\cos t \cancel{+} - C\sin t \in V \text{; so,}$$

$$T(y) = y'' - y \in V.$$

(3.) The operator $T: V \to V$ is 1-to-1. Indeed, we need to assume that $y \in V$ and $T(y) = 0$; prove that $y \equiv 0$. But $T(y) = 0 \Rightarrow y'' - y = 0$; $\Rightarrow$ y is ~~the~~ a solution of the homogeneous equation $\Rightarrow$ y has to have the form $c_1 e^{t} + c_2 e^{-t}$, which is impossible if $y \neq 0$, $y \in V$. (Alternatively, calculate $y'' - y$ for $y = Ae^{-2t} + B\cos t + C\sin t$ and prove that if $y'' - y = 0$, then $A = B = C = 0$.)

(4.) V is a finite dimensional space, $T: V \to V$ is 1-to-1 $\Rightarrow$ by the IMT, T has to be onto. Since $f = 5e^{-2t} + \cos t \in V$, there exists $y \in V$ such that $T(y) = \cancel{y''}\ f$. But this means that y solves the equation

$$y'' - y = f.$$