Math 54, Section 214 Your name: Key
Quiz 7, March 19, 2010

Please write your name **on each sheet**. Show your work clearly and in order, including intermediate steps in the solutions and the final answer.

1. (7 pt) Can the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

be represented as $A = PDP^{-1}$, where D is diagonal and P is invertible? If so, find P and D.

Bonus (no points, hard): what are all possible values of P and D?

1. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} =$

$= (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 4\lambda + 3) =$

$= (1-\lambda)^2(3-\lambda) \rightarrow$ eigenvalues $\lambda = \underset{(\text{mult. } 2)}{1}, 3.$

2. $\text{Nul}(A - 1 \cdot I) = \text{Nul}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Nul}\begin{bmatrix} \boxed{1} & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$

Basis for that: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. $\text{Nul}(A - 3 \cdot I) = \text{Nul}\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & -2 \end{bmatrix} = \text{Nul}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};$

Basis for that: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

4. A has a basis of eigenvectors $\rightarrow$ it is diagonalisable.

We can take $P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$

2. (7 pt) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix in the standard basis is

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ so that the $\mathcal{B}$-matrix of the transformation T has the form

$$\lambda \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Find the numbers λ and ϕ.

(1) $|A-\lambda I| = \begin{vmatrix} -\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = (\lambda-1)^2+1;$

eigenvalues $1 \pm i$.

(2) $\text{Nul}\,(A-(1-i)I) = \text{Nul} \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix};$

a complex basis of that is $\vec{v} = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}.$

(3) We can form the basis $\mathcal{B}$ out of the real and imaginary parts of $\vec{v}$: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$

The $\mathcal{B}$-matrix for ~~A~~ T is

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \lambda \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$, when $\lambda = \sqrt{2}$ and $\phi = \pi/4$.

<u>Note</u>: there are multiple correct answers.

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3. (6 pt) Let W be the subspace of $\mathbb{R}^2$ spanned by the vector

$$\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find a basis for $W^\perp$.

Let $\vec{x} = \begin{bmatrix} \vec{x}_1 \\ x_2 \end{bmatrix}$. Then $\vec{x} \perp \vec{w} \iff$

$\iff 2x_1 + x_2 = 0$. So, $W^\perp = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, 2x_1 + x_2 = 0 \right\}$;

a basis of that is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

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