Math 54, Section 214 Your name: Key

Quiz 6, March 12, 2010

Please write your name on each sheet. Show your work clearly and in order, including intermediate steps in the solutions and the final answer.

1. (7 pt) Assume that V is a two-dimensional vector space and we are given two bases $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ of V such that

$$\vec{b}_1 = 2(\vec{c}_1 - \vec{c}_2), \ \vec{b}_2 = 4\vec{c}_2 - 3\vec{c}_1.$$

(a) Find the change-of-coordinates matrices from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

(b) Assume that $\vec{x}$ is a vector in V whose $\mathcal{C}$-coordinate vector is

$$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find the $\mathcal{B}$-coordinate vector of $\vec{x}$.

(a) $$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[[\vec{b}_1]_{\mathcal{C}} \ [\vec{b}_2]_{\mathcal{C}} \right] = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix};$$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \right)^{-1} = \frac{1}{2}\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 \\ 1 & 1 \end{bmatrix}.$$

(b) $$[\vec{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} [\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 2 & 3/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ 3 \end{bmatrix}$$

2. (6 pt) Find the characteristic polynomial and all eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}.$$

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} =$$

$$= \lambda(\lambda - 3) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4).$$

The eigenvalues are $-1, 4$.

Math 54, Section 214 Your name: Key

3. (7 pt) Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of A and their algebraic multiplicities.

(b) For each eigenvalue of A, find a basis of the corresponding eigenspace.

(a) A is upper triangular $\rightarrow$

$\rightarrow$ the eigenvalues are on the diagonal:

2 (multiplicity 1), 3 (multiplicity 2)

(b) $\underline{\lambda = 2}$: $A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$,

Basis for $\text{Nul}(A - \lambda I)$ is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

$\underline{\lambda = 3}$: $A - \lambda I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$

Basis for $\text{Nul}(A - \lambda I)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.