DIFFERENTIAL OPERATORS ON EUCLIDEAN SPACES

SEMYON DYATLOV

ABSTRACT. These are notes for Lecture 2 of the Math 279 course 'Semiclassical Analysis' taught at UC Berkeley in Fall 2018. Caution: there are some small notational differences with [Zw], in particular the latter uses Weyl quantization.

1. The nonsemiclassical case

1.1. **Notation.** We will work on $\mathbb{R}^n$, denoting its elements by $x = (x_1, \dots, x_n)$. We use the following notation:

$$\partial_{x_j} := \frac{\partial}{\partial x_j}, \quad D_{x_j} := \frac{1}{i}\partial_{x_j}.$$

The notation D_{x_j} is different from many other PDE texts however it will be very useful in our course.

For higher order derivatives we use multiindex notation. A *multiindex* has the form

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad \alpha_j \in \mathbb{N}_0.$$

Its *order* is defined by

$$|\alpha| := \alpha_1 + \dots + \alpha_n.$$

For a multiindex α, we define the corresponding operators

$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$$

and the function

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

We also define

$$\alpha! := \alpha_1! \cdots \alpha_n!.$$

For two multiindices α, β, we say

$$\alpha \le \beta \quad \text{if} \quad \alpha_j \le \beta_j \quad \text{for all } j$$

in which case $\beta - \alpha$ is also a multiindex.

1.2. **Differential operators.** Our differential operators will have coefficients in the following space:

Definition 1.1. *A function $a \in C^\infty(\mathbb{R}^n;\mathbb{C})$ is called* ***nice*** *if each derivative $\partial_x^\alpha a$ is uniformly bounded in x:*

$$\sup_{x\in\mathbb{R}^n} |\partial_x^\alpha a(x)| < \infty. \tag{1.1}$$

We denote the space of all nice functions by $S(\mathbb{R}^n)$ and endow it with the countably many seminorms (1.1).

Exercise 1. *Show that $S(\mathbb{R}^n)$ is a* ***Fréchet space****: if we put*

$$\|a\|_k := \sum_{|\alpha|=k} \sup |\partial_x^\alpha a|$$

then the metric $d(\bullet,\bullet)$ on $S(\mathbb{R}^n)$ defined by

$$d(a,b) := \sum_{k\geq 0} 2^{-k} \frac{\|a-b\|_k}{1+\|a-b\|_k}$$

makes $S(\mathbb{R}^n)$ a complete metric space.

We can now define differential operators:

Definition 1.2. *Let $m \in \mathbb{N}_0$. A* ***differential operator*** *of order m is an operator on $C^\infty(\mathbb{R}^n)$ of the form*

$$A = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha \tag{1.2}$$

for some functions $a_\alpha(x) \in S(\mathbb{R}^n)$. Denote by $\operatorname{Diff}^m(\mathbb{R}^n)$ the space of all differential operators of order m.

Differential operators have the following properties:

Proposition 1.3. *We have*

1. $\operatorname{Diff}^m(\mathbb{R}^n) \subset \operatorname{Diff}^{m+1}(\mathbb{R}^n)$ *for all $m \geq 0$;*
2. *the identity operator I lies in $\operatorname{Diff}^0(\mathbb{R}^n)$;*
3. *if $A \in \operatorname{Diff}^m(\mathbb{R}^n)$ and $B \in \operatorname{Diff}^\ell(\mathbb{R}^n)$, then $AB \in \operatorname{Diff}^{m+\ell}(\mathbb{R}^n)$;*
4. *if $A \in \operatorname{Diff}^m(\mathbb{R}^n)$ then there exists $A^* \in \operatorname{Diff}^m(\mathbb{R}^n)$ which is the formal adjoint of A, that is*

$$\langle Au, v\rangle_{L^2(\mathbb{R}^n)} = \langle u, A^* v\rangle_{L^2(\mathbb{R}^n)} \quad \text{for all} \quad u \in C^\infty(\mathbb{R}^n),\ v \in C_c^\infty(\mathbb{R}^n).$$

Exercise 2. *Use integration by parts to show that if A is given by (1.2) then*

$$A^* = \sum_{|\alpha|\leq m} D_x^\alpha\, \overline{a_\alpha(x)},$$

by which we mean

$$A^*u(x) = \sum_{|\alpha|\leq m} D_x^\alpha(\overline{a_\alpha(x)}u(x)).$$

1.3. **Full symbol of an operator.** To a differential operator $A \in \mathrm{Diff}^m(\mathbb{R}^n)$ we associate its full symbol which is a function $a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ defined as follows:

Definition 1.4. *Let $A \in \mathrm{Diff}^m(\mathbb{R}^n)$ be given by (1.2). Define the **full symbol** of A as the following polynomial in $\xi \in \mathbb{R}^n$ with coefficients in $x \in \mathbb{R}^n$:*

$$a(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{1.3}$$

*Conversely if $a(x,\xi)$ is given by (1.3) then define the **quantization** $\mathrm{Op}(a) \in \mathrm{Diff}^m(\mathbb{R}^n)$ of a by the formula (1.2).*

Define the space

$$\mathrm{Poly}^m(\mathbb{R}^{2n}) \subset C^\infty(\mathbb{R}^{2n})$$

consisting of functions of the form (1.3) with each $a_\alpha(x)$ lying in $S(\mathbb{R}^n)$. (It is a Fréchet space similarly to Exercise 1, taking the $S(\mathbb{R}^n)$ seminorms of each coefficient a_α.) Then we have a linear isomorphism

$$\mathrm{Op} : \mathrm{Poly}^m(\mathbb{R}^{2n}) \to \mathrm{Diff}^m(\mathbb{R}^n).$$

Here are a few examples of quantization of symbols:

- if $a(x)$ is a function of x only, then $\mathrm{Op}(a)$ is the multiplication operator by a, namely $\mathrm{Op}(a)u(x) = a(x)u(x)$;
- if $a(x) = \xi_j$, then $\mathrm{Op}(a) = D_{x_j}$;
- if $a(x) = |\xi|^2 + V(x)$ where $V \in S(\mathbb{R}^n)$ is a potential, then $\mathrm{Op}(a) = -\Delta + V$ is a Schrödinger operator.

Remark 1.5. *Formally we could write*

$$\mathrm{Op}(a) = a(x, D_x).$$

However, this notation can be misleading since the operators x and D_x do not commute. The quantization Op that we use, given by

$$\mathrm{Op}(a) = \sum_{|\alpha|\leq m} a_\alpha D_x^\alpha$$

*is often called the **standard** or **left** quantization. There is also the **right** quantization*

$$\mathrm{Op}^1(a) = \sum_{|\alpha|\leq m} D_x^\alpha a_\alpha$$

and many intermediate choices such as the ***Weyl quantization*** Op^{w} *used in* [Zw]. *For instance, if* $n = 1$ *and* $a(x, \xi) = x\xi$ *(ignoring the fact that* $x \notin S(\mathbb{R})$*) then*

$$\mathrm{Op}(a) = xD_x, \quad \mathrm{Op}^1(a) = D_x x = xD_x - i, \quad \mathrm{Op}^{\mathrm{w}}(a) = xD_x - \tfrac{i}{2}.$$

On manifolds the situation is even worse since there is no canonical quantization procedure. However, the multitude of existing quantizations does not cause a problem for the theory because:

(1) *every choice of quantization gives the same class of operators* Diff^m*;*
(2) *and the notion of the* ***principal*** *symbol of an operator (see §1.4) is independent of quantization.*

Coming back to the standard quantization, one can recover the symbol from an operator by the following

Lemma 1.6 (Oscillatory testing)**.** *For each* $\xi \in \mathbb{R}^n$ *define the function*

$$e_\xi \in C^\infty(\mathbb{R}^n), \quad e_\xi(x) = e^{i\langle x, \xi\rangle}.$$

Then for each $a \in \mathrm{Poly}^m(\mathbb{R}^{2n})$ *and* $\xi \in \mathbb{R}^n$ *we have*

$$(\mathrm{Op}(a)e_\xi)(x) = a(x, \xi)e_\xi(x). \tag{1.4}$$

Proof. We have for each multiindex α

$$D_x^\alpha e_\xi(x) = \xi^\alpha e_\xi(x)$$

from which (1.4) follows immediately. This explains the use of $D_x = \frac{1}{i}\partial_x$ in the definition (1.2). □

The full symbol of an operator has several nice algebraic properties. The first one is the following

Proposition 1.7 (Product Rule for the full symbol)**.** *Let* $a \in \mathrm{Poly}^m(\mathbb{R}^{2n})$*,* $b \in \mathrm{Poly}^\ell(\mathbb{R}^{2n})$*. Then*

$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(a\#b)$$

where $a\#b \in \mathrm{Poly}^{m+\ell}(\mathbb{R}^{2n})$ *is defined by the following finite sum:*

$$a\#b(x, \xi) = \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a(x, \xi))(\partial_x^\alpha b(x, \xi)). \tag{1.5}$$

In particular

$$a\#b = ab - i\sum_{j=1}^n (\partial_{\xi_j} a)(\partial_{x_j} b) + \mathrm{Poly}^{m+\ell-2}(\mathbb{R}^{2n}). \tag{1.6}$$

Proof. First of all, we note that since a is a polynomial of order m in ξ, $\partial_\xi^\alpha a$ is a polynomial of order $m-|\alpha|$. In particular, $\partial_\xi^\alpha a = 0$ when $|\alpha| \geq m$, thus the sum (1.5) indeed has only finitely many nonzero terms. This also shows that (1.5) implies (1.6), as each term in (1.5) lies in $\mathrm{Poly}^{m+\ell-|\alpha|}(\mathbb{R}^{2n})$.

To show (1.5), we first note that Proposition 1.3 implies that $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c)$ for some $c \in \mathrm{Poly}^{m+\ell}(\mathbb{R}^{2n})$. To show that $c = a\#b$, without loss of generality we assume that $a(x,\xi) = a_\gamma(x)\xi^\gamma$ for some multiindex γ, $|\gamma| \leq m$. Using oscillatory testing, Lemma 1.6, we have for each $\xi \in \mathbb{R}^n$

$$\mathrm{Op}(c)e_\xi = \mathrm{Op}(a)\,\mathrm{Op}(b)e_\xi = \mathrm{Op}(a)(b(\bullet,\xi)e_\xi) = a_\gamma(x)D_x^\gamma(b(x,\xi)e_\xi(x)).$$

Using the product rule for derivatives we compute

$$\begin{aligned}\mathrm{Op}(c)e_\xi &= \sum_{\alpha\leq\gamma}\frac{\gamma!}{\alpha!(\gamma-\alpha)!}a_\gamma(x)\big(D_x^\alpha b(x,\xi)\big)\big(D_x^{\gamma-\alpha}e_\xi(x)\big)\\ &= \sum_{\alpha\leq\gamma}\frac{\gamma!}{\alpha!(\gamma-\alpha)!}a_\gamma(x)\big(D_x^\alpha b(x,\xi)\big)\xi^{\gamma-\alpha}e_\xi(x).\end{aligned}$$

Applying Lemma 1.6 again, we see that $(\mathrm{Op}(c)e_\xi)(x) = c(x,\xi)e_\xi(x)$ and thus

$$c(x,\xi) = \sum_{\alpha\leq\gamma}a_\gamma(x)\frac{\gamma!}{\alpha!(\gamma-\alpha)!}\big(D_x^\alpha b(x,\xi)\big)\xi^{\gamma-\alpha}. \tag{1.7}$$

On the other hand, recalling that $a(x,\xi) = a_\gamma(x)\xi^\gamma$, we have

$$\partial_\xi^\alpha a(x,\xi) = \begin{cases} a_\gamma(x)\frac{\gamma!}{(\gamma-\alpha)!}\xi^{\gamma-\alpha}, & \alpha\leq\gamma;\\ 0, & \text{otherwise}\end{cases}$$

therefore

$$a\#b(x,\xi) = \sum_{\alpha\leq\gamma}a_\gamma(x)\frac{\gamma!}{\alpha!(\gamma-\alpha)!}\xi^{\gamma-\alpha}\big(D_x^\alpha b(x,\xi)\big). \tag{1.8}$$

Comparing (1.7) and (1.8) we see that $c = a\#b$ as needed. □

Example 1.8. *We use the following example to illustrate Proposition 1.7:*

$$a = |\xi|^2 = \sum_{j=1}^n \xi_j^2, \quad \mathrm{Op}(a) = -\Delta; \quad b = b(x), \quad \mathrm{Op}(b) = b.$$

A direct computation shows that

$$\mathrm{Op}(a)\,\mathrm{Op}(b)u = -\Delta(bu) = -b(\Delta u) - 2\langle\nabla b,\nabla u\rangle - (\Delta b)u = \mathrm{Op}(c)u$$

where

$$c(x,\xi) = -|\xi|^2 b(x) - 2i\sum_{j=1}^n \xi_j(\partial_{x_j}b(x)) - \Delta b(x). \tag{1.9}$$

We see from (1.5) that indeed $c = a\# b$. Indeed, the first term on the right-hand side of (1.9) comes from the case $|\alpha| = 0$, the second term comes from the case $|\alpha| = 1$, and the third term comes from the case $|\alpha| = 2$.

The other algebraic property is the following formula for the symbol of the formal adjoint, introduced in Proposition 1.3.

Proposition 1.9 (Adjoint Rule for the full symbol)**.** *Let $a \in \operatorname{Poly}^m(\mathbb{R}^{2n})$. Then*

$$\operatorname{Op}(a)^* = \operatorname{Op}(a^*)$$

where $a^* \in \operatorname{Poly}^m(\mathbb{R}^{2n})$ is defined by the following finite sum:

$$a^*(x,\xi) = \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi)}. \tag{1.10}$$

In particular

$$a^* = \bar{a} + \operatorname{Poly}^{m-1}(\mathbb{R}^{2n}). \tag{1.11}$$

Proof. As in Proposition 1.7 it suffices to consider the case when $a(x,\xi) = a_\gamma(x)\xi^\gamma$. From Exercise 2 we have

$$\operatorname{Op}(a)^* = D_x^\gamma \overline{a_\gamma(x)} = \operatorname{Op}(\xi^\gamma) \operatorname{Op}(\overline{a_\gamma(x)}).$$

Applying Proposition 1.7 we see that

$$\operatorname{Op}(a)^* = \operatorname{Op}(b) \quad \text{where} \quad b(x,\xi) = \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha(\xi^\gamma))(\partial_x^\alpha \overline{a_\gamma(x)})$$

It is straightforward to verify that $b = a^*$ where a^* is given by (1.10). □

1.4. **The principal symbol.** The Product Rule and Adjoint Rule for the full symbols presented above are complicated, and the full symbol calculus would become even more complicated if we replaced $\mathbb{R}^n$ with an arbitrary manifold. However, for many purposes it is enough to know the leading order part of the symbol, for which the algebra becomes much simpler. In this section we introduce this leading part, called the *principal symbol*.

Define

$$\operatorname{HPoly}^m(\mathbb{R}^{2n}) \subset \operatorname{Poly}^m(\mathbb{R}^{2n})$$

as the set of $a(x,\xi) \in \operatorname{Poly}^m(\mathbb{R}^{2n})$ which are *homogeneous* polynomials of order m in ξ, that is they have the form (1.3) with the sum over multiindices α such that $|\alpha| = m$. Then the principal symbol is given by

Definition 1.10. *Let* $A \in \mathrm{Diff}^m(\mathbb{R}^n)$ *and let* a *be its full symbol, so that* $A = \mathrm{Op}(a)$*. Define the* ***principal symbol*** $\sigma_m(A) \in \mathrm{HPoly}^m(\mathbb{R}^{2n})$ *as the order* m *part of* a*. That is, if* a *is given by* (1.3) *then*

$$\sigma_m(A)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

Note that we make m explicit in the notation σ_m, since for instance $\sigma_1(D_{x_1}) = \xi_1$ but $\sigma_2(D_{x_1}) = 0$. However, most of the time the principal symbol is denoted by $\sigma(A)$ when it is clear what the natural value of the order m is.

It follows immediately from Definition 1.10 that the principal symbol has the following properties:

Proposition 1.11. *Let* $m \in \mathbb{N}_0$*. Then:*

(1) *For* $A \in \mathrm{Diff}^m(\mathbb{R}^n)$ *we have* $\sigma_m(A) = 0$ *if and only if* $A \in \mathrm{Diff}^{m-1}(\mathbb{R}^n)$ *(with the convention* $\mathrm{Diff}^{-1}(\mathbb{R}^n) := \{0\}$*).*
(2) *The map* $\sigma_m : \mathrm{Diff}^m(\mathbb{R}^n) \to \mathrm{HPoly}^m(\mathbb{R}^{2n})$ *is onto. In fact* $\sigma_m(\mathrm{Op}(a)) = a$ *for every* $a \in \mathrm{HPoly}^m(\mathbb{R}^{2n})$*.*

A more abstract way to state Proposition 1.11 is that the following sequence is exact:

$$0 \longrightarrow \mathrm{Diff}^{m-1}(\mathbb{R}^n) \xrightarrow{\iota_m} \mathrm{Diff}^m(\mathbb{R}^n) \xrightarrow{\sigma_m} \mathrm{HPoly}^m(\mathbb{R}^{2n}) \longrightarrow 0$$

where ι_m denotes the inclusion operator.

From Propositions 1.7 and 1.9 we immediately deduce algebraic rules for principal symbols:

Proposition 1.12 (Principal symbol calculus)**.** *Assume that* $A \in \mathrm{Diff}^m(\mathbb{R}^n)$ *and* $B \in \mathrm{Diff}^\ell(\mathbb{R}^n)$*. Then we have*

(1) *Product Rule:* $AB \in \mathrm{Diff}^{m+\ell}(\mathbb{R}^n)$ *and*
$$\sigma_{m+\ell}(AB) = \sigma_m(A)\sigma_\ell(B); \tag{1.12}$$
(2) *Commutator Rule: the commutator* $[A,B] := AB - BA$ *lies in* $\mathrm{Diff}^{m+\ell-1}(\mathbb{R}^n)$ *and*
$$\sigma_{m+\ell-1}([A,B]) = -i\{\sigma_m(A), \sigma_\ell(B)\} \tag{1.13}$$
where $\{\bullet,\bullet\}$ *denotes the Poisson bracket:*
$$\{a,b\} := \sum_{j=1}^n (\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b);$$
(3) *Adjoint Rule:* $A^* \in \mathrm{Diff}^m(\mathbb{R}^n)$ *and*
$$\sigma_m(A^*) = \overline{\sigma_m(A)}. \tag{1.14}$$

The Commutator Rule is new to us and it has profound consequences for the theory. Note that the first part of this rule, namely $[A, B] \in \mathrm{Diff}^{m+\ell-1}(\mathbb{R}^n)$, actually follows from the Product Rule (1.12) since $\sigma_{m+\ell}([A, B]) = 0$. The formula (1.13) is where the relation with Hamiltonian dynamics comes in. Indeed, if a is real-valued then we have

$$\{a, b\} = H_a b$$

where $H_a = \sum_{j=1}^n (\partial_{\xi_j} a)\partial_{x_j} - (\partial_{x_j} a)\partial_{\xi_j}$ is the *Hamilton vector field* of a. The flow of this field,

$$e^{tH_a} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad t \in \mathbb{R},$$

is the one-parameter group defined by $e^{tH_a}(x_0, \xi_0) = (x(t), \xi(t))$ where $(x(t), \xi(t))$ solves the initial value problem for *Hamilton's equations*

$$\begin{aligned} \dot{x}_j(t) = \partial_{\xi_j} a(x(t), \xi(t)), \quad \dot{\xi}_j(t) = -\partial_{x_j} a(x(t), \xi(t)),\\ x(0) = x_0, \quad \xi(0) = \xi_0. \end{aligned}$$

Please see [Zw, §§2.1,2.2] and the beginning of [Zw, §2.4] for more on Hamilton vector fields.

Exercise 3 (for students familiar with Riemannian geometry)**.** *Let g be a Riemannian metric on $\mathbb{R}^n$:*

$$g = \sum_{j,\ell=1}^n g_{j\ell}(x)\, dx_j dx_\ell.$$

For simplicity assume that g is the Euclidean metric for x outside a compact set. Let $\Delta_g \in \mathrm{Diff}^2(\mathbb{R}^n)$ be the Laplace–Beltrami operator induced by g.

(1) *Show that $\sigma_2(-\Delta_g) = p$ where*

$$p(x, \xi) = \sum_{j,\ell=1}^n g^{j\ell}(x)\xi_j\xi_\ell, \quad (g^{j\ell}(x)) = (g_{j\ell}(x))^{-1}.$$

(2) *Show that the Hamiltonian flow e^{tH_p} is related to the geodesic flow of g as follows: if $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and*

$$(x(t), \xi(t)) := e^{tH_p}(x_0, \xi_0)$$

then $x(t)_{t\in\mathbb{R}}$ is a geodesic on $(\mathbb{R}^n, g)$ and $\xi(t)$ is related to the tangent vector $\dot{x}(t)$ as follows:

$$2\xi_j(t) = \sum_{\ell=1}^n g_{j\ell}(x(t))\dot{x}_\ell(t).$$

2. The semiclassical case

2.1. **Definition.** We now consider the more general class of *semiclassical* differential operators. These are operators of the form $\mathrm{Op}_h(a)$, where the quantization procedure Op is altered to put h next to each ∂_{x_j}:

$$\text{if}\quad a(x,\xi)=\sum_{|\alpha|\le m}a_\alpha(x)\xi^\alpha\quad\text{then}\quad \mathrm{Op}_h(a):=\sum_{|\alpha|\le m}a_\alpha(x)(hD_x)^\alpha \tag{2.1}$$

where $D_x=\frac{1}{i}\partial_x$ as before and $(hD_x)^\alpha:=h^{|\alpha|}D_x^\alpha$. So for instance

$$\mathrm{Op}_h(|\xi|^2-1)=-h^2\Delta-1.$$

The resulting operator $\mathrm{Op}_h(a)$ is now a family of operators which depends on the *semiclassical parameter* $h>0$. We will often study the *semiclassical limit* $h\to 0$ but one could also take a fixed value of h: say for $h=1$ we have $\mathrm{Op}_h=\mathrm{Op}$ and we recover the nonsemiclassical theory. We will always assume $h\in(0,1]$ (since the 'large h' limit is irrelevant in this course). The functions we study will often oscillate at frequencies $\sim h^{-1}$ (see e.g. Lemma 2.2 and Exercise 4). For these functions hD_x has the same strength as the identity operator, so it is illegal to just put $h:=0$ in (2.1).

Even if we start with h-independent symbols a, algebraic operations lead to symbols dependent on h. For instance (ignoring again that $x\notin S(\mathbb{R})$)

$$\mathrm{Op}_h(\xi)\,\mathrm{Op}_h(x)=(hD_x)x=x(hD_x)-ih=\mathrm{Op}_h(x\xi-ih).$$

Thus we introduce h-dependent symbols:

Definition 2.1. *We say a family of functions $a(x;h)\in C_x^\infty(\mathbb{R}^n)$, depending on $h\in(0,1]$, lies in the class $S_h(\mathbb{R}^n)$, if for every multiindex α*

$$\sup_{h\in(0,1]}\sup_{x\in\mathbb{R}^n}|\partial_x^\alpha a(x;h)|<\infty. \tag{2.2}$$

For $m\in\mathbb{N}_0$ we define the class $\mathrm{Poly}_h^m(\mathbb{R}^{2n})$ consisting of all functions $a(x,\xi;h)$ of the form

$$a(x,\xi;h)=\sum_{|\alpha|\le m}a_\alpha(x;h)\xi^\alpha$$

where each $a_\alpha(x;h)$ lies in $S_h(\mathbb{R}^n)$.

Note that similarly to Exercise 1, $S_h(\mathbb{R}^n)$ is a Fréchet space with respect to the seminorms (2.2), and similarly $\mathrm{Poly}_h^m(\mathbb{R}^{2n})$ is a Fréchet space. Note also that we do not require any smoothness of $a(x;h)$ with respect to h, only bounds which are uniform in h.

Now define the class of *semiclassical differential operators* $\mathrm{Diff}_h^m(\mathbb{R}^n)$ as follows:

$$\mathrm{Diff}_h^m(\mathbb{R}^n):=\{\mathrm{Op}_h(a)\mid a\in\mathrm{Poly}_h^m(\mathbb{R}^{2n})\}.$$

2.2. **Full symbol calculus.** We now give the semiclassical analogues of the algebraic properties in §1.3. These are established in a very similar way to the nonsemiclassical version so we omit the proofs. First of all, oscillatory testing is given by

Lemma 2.2. *For each $\xi \in \mathbb{R}^n$ and $h > 0$ define the function*

$$e_{\xi,h} \in C^\infty(\mathbb{R}^n), \quad e_{\xi,h}(x) = e^{i\langle x,\xi\rangle/h}.$$

Then for each $a \in \operatorname{Poly}_h^m(\mathbb{R}^{2n})$, $\xi \in \mathbb{R}^n$, and $h \in (0,1]$ we have

$$(\operatorname{Op}_h(a)e_{\xi,h})(x) = a(x,\xi;h)e_{\xi,h}(x).$$

Next, the algebraic properties are given by

Proposition 2.3 (Calculus for the full symbol)**.** *Let $a \in \operatorname{Poly}_h^m(\mathbb{R}^{2n})$, $b \in \operatorname{Poly}_h^\ell(\mathbb{R}^{2n})$. Then*

$$\operatorname{Op}_h(a)\operatorname{Op}_h(b) = \operatorname{Op}_h(a\#b), \quad \operatorname{Op}_h(a)^* = \operatorname{Op}_h(a^*)$$

where $a\#b \in \operatorname{Poly}_h^{m+\ell}(\mathbb{R}^{2n})$, $a^* \in \operatorname{Poly}_h^m(\mathbb{R}^{2n})$ are defined by the finite sums

$$a\#b(x,\xi;h) = \sum_\alpha \frac{(-ih)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha a(x,\xi;h))(\partial_x^\alpha b(x,\xi;h)), \tag{2.3}$$

$$a^*(x,\xi;h) = \sum_\alpha \frac{(-ih)^{|\alpha|}}{\alpha!}\partial_x^\alpha\partial_\xi^\alpha\overline{a(x,\xi;h)}. \tag{2.4}$$

Remark 2.4. *The terms with $|\alpha| = k$ in (2.3) lie in $h^k\operatorname{Poly}^{m+\ell-k}(\mathbb{R}^{2n})$, that is each next term gains a power of h as well as lowers the degree of the polynomial in ξ. Similarly the terms with $|\alpha| = k$ in (2.4) lie in $h^k\operatorname{Poly}^{m-k}(\mathbb{R}^{2n})$.*

2.3. **Principal symbol calculus.** We finally introduce the notion of semiclassical principal symbol, generalizing the nonsemiclassical one defined in §1.4. By Remark 2.4, each term in the expansions (2.3) and (2.4) improves both in powers of h and in the degree of the polynomial. It thus makes sense to define the principal symbol as an element in the quotient space

$$\frac{\operatorname{Poly}_h^m(\mathbb{R}^{2n})}{h\operatorname{Poly}_h^{m-1}(\mathbb{R}^{2n})}. \tag{2.5}$$

More precisely, for $A = \operatorname{Op}_h(a) \in \operatorname{Diff}_h^m(\mathbb{R}^n)$ we define the semiclassical principal symbol $\sigma_{m,h}(A)$ as the equivalence class $[a]$ of a in (2.5). Similarly to §1.4 we have the following

Proposition 2.5 (Semiclassical principal symbol calculus)**.** *Let $m \in \mathbb{N}_0$. Then:*

1. *For $A \in \operatorname{Diff}_h^m(\mathbb{R}^n)$ we have $\sigma_{m,h}(A) = 0$ if and only if $A \in h\operatorname{Diff}_h^{m-1}(\mathbb{R}^n)$.*
2. *The map*

$$\sigma_m : \operatorname{Diff}_h^m(\mathbb{R}^n) \to \frac{\operatorname{Poly}_h^m(\mathbb{R}^{2n})}{h\operatorname{Poly}_h^{m-1}(\mathbb{R}^{2n})}$$

is onto. In fact, for each $a \in \operatorname{Poly}_h^m(\mathbb{R}^n)$ we have $\sigma_m(\operatorname{Op}_h(a)) = [a]$.

(3) *Product Rule: if* $A \in \mathrm{Diff}_h^m(\mathbb{R}^n)$,$B \in \mathrm{Diff}_h^\ell(\mathbb{R}^n)$ *then* $AB \in \mathrm{Diff}_h^{m+\ell}(\mathbb{R}^n)$ *and*

$$\sigma_{m+\ell,h}(AB) = \sigma_{m,h}(A)\sigma_{\ell,h}(B). \tag{2.6}$$

(4) *Commutator Rule: if* $A \in \mathrm{Diff}_h^m(\mathbb{R}^n)$, $B \in \mathrm{Diff}_h^\ell(\mathbb{R}^n)$ *then* $[A,B] \in h\,\mathrm{Diff}_h^{m+\ell-1}(\mathbb{R}^n)$ *and*

$$\sigma_{m+\ell-1,h}(h^{-1}[A,B]) = -i\{\sigma_{m,h}(A), \sigma_{\ell,h}(B)\}. \tag{2.7}$$

(5) *Adjoint Rule: if* $A \in \mathrm{Diff}_h^m(\mathbb{R}^n)$ *then* $A^* \in \mathrm{Diff}_h^m(\mathbb{R}^n)$ *and*

$$\sigma_{m,h}(A^*) = \overline{\sigma_{m,h}(A)}. \tag{2.8}$$

Remark 2.6. *Note that for fixed* h*, the space* (2.5) *would turn into* $\mathrm{Poly}_h^m(\mathbb{R}^{2n})/\,\mathrm{Poly}_h^{m-1}(\mathbb{R}^{2n})$ *which is canonically isomorphic to the space of homogeneous polynomials* $\mathrm{HPoly}_h^m(\mathbb{R}^{2n})$ *used in the definition of the nonsemiclassical principal symbol in §1.4.*

Remark 2.7. *In practice we use the notation* $\sigma_h(A)$ *when it is clear which value of* m *is meant. Abusing the notation even more, if we know that* $A \in h^k\,\mathrm{Diff}_h^m(\mathbb{R}^n)$ *we may write* $\sigma_h(A) := h^k\sigma_{m,h}(h^{-k}A)$*. Under this notation the Commutator Rule above becomes*

$$\sigma_h([A,B]) = -ih\{\sigma_h(A), \sigma_h(B)\}.$$

Remark 2.8. *An equivalent way to write* (2.6)–(2.8) *is: for* $a \in \mathrm{Poly}_h^m(\mathbb{R}^{2n})$ *and* $b \in \mathrm{Poly}_h^\ell(\mathbb{R}^{2n})$,

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + h\,\mathrm{Diff}_h^{m+\ell-1}(\mathbb{R}^n), \tag{2.9}$$

$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = -ih\,\mathrm{Op}_h(\{a,b\}) + h^2\,\mathrm{Diff}_h^{m+\ell-2}(\mathbb{R}^n), \tag{2.10}$$

$$\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\overline{a}) + h\,\mathrm{Diff}_h^{m-1}(\mathbb{R}^n). \tag{2.11}$$

Remark 2.9. *In* [Zw, Chapter 4] *the principal symbol is defined modulo* hS^m *(with* S^m *denoting the space of symbols of order* m*) rather than modulo* hS^{m-1}*, that is we gain a power of* h *but no powers of* ξ*. This is related to the more general class of symbols used in* [Zw, §4.4.1] *which do not gain powers of* ξ *when differentiated (unlike polynomials which lose one power with each differentiation). The classes with* hS^{m-1} *remainder are introduced later in* [Zw, §9.3.1] *and are essential for defining calculus on manifolds.*

Example 2.10. *For the Schrödinger operator*

$$P = -h^2\Delta + V(x)$$

its semiclassical principal symbol is

$$\sigma_h(P) = [p] \quad \textit{where} \quad p(x,\xi) = |\xi|^2 + V(x).$$

That is, semiclassically the potential has the same order of magnitude as the Laplacian. This is different from the nonsemiclassical case, where $\sigma(-\Delta + V(x)) = |\xi|^2$*, so the potential is a perturbation.*

Note that if we wanted to make $-\Delta+V$ semiclassical by rescaling, we would arrive to $-h^2\Delta+h^2V$ which has semiclassical principal symbol $|\xi|^2$. Conversely if we wanted to make $-h^2\Delta+V$ nonsemiclassical we would arrive to $-\Delta+h^{-2}V$, that is the potential V is multiplied by a large coupling constant.

Remark 2.11. *If the principal symbol $\sigma_h(P)$ has a representative p which is h-independent as in the above example, we often just write $\sigma_h(P)=p$. This is the case in most applications. However, in general we might not have an h-independent representative: examples are given by*

$$P=-(h^2+h^3)\Delta\in\mathrm{Diff}_h^2(\mathbb{R}^n);\quad P=e^{i/h}\in\mathrm{Diff}_h^0(\mathbb{R}^n).$$

Note that in the first case we cannot write $\sigma_h(P)=[|\xi|^2]$ since the full symbol is $(1+h)|\xi|^2$ and $h|\xi|^2\notin h\,\mathrm{Diff}_h^1(\mathbb{R}^n)$ (it only lies in $h\,\mathrm{Diff}_h^2(\mathbb{R}^n)$).

The next exercise generalizes oscillatory testing, Lemma 2.2 (think of taking $\Phi(x):=\langle x,\xi\rangle$ for fixed ξ). It could be used to understand how the principal symbol changes under diffeomorphisms and thus how to define the principal symbol invariantly on manifolds (see §2.4 below).

Exercise 4. *Take two functions*

$$\Phi\in C^\infty(\mathbb{R}^n;\mathbb{R}),\quad b\in C_{\mathrm{c}}^\infty(\mathbb{R}^n;\mathbb{C})$$

and consider

$$u(x;h):=e^{i\Phi(x)/h}b(x).$$

We often call Φ the ***phase*** *of u and b its* ***amplitude****. Let $a\in\mathrm{Poly}^m(\mathbb{R}^{2n})$ be h-independent (for simplicity). Show that*

$$\mathrm{Op}_h(a)u=e^{i\Phi(x)/h}c(x;h),\quad c(x;h)=\sum_{k=0}^m h^k c_k(x)$$

where $c_k\in C_{\mathrm{c}}^\infty(\mathbb{R}^n;\mathbb{C})$ are h-independent and

$$c_0(x)=a(x,\nabla\Phi(x))b(x),$$
$$c_1(x)=-i\sum_{j=1}^n(\partial_{\xi_j}a(x,\nabla\Phi(x)))\partial_{x_j}b(x)-\frac{i}{2}\sum_{j,k=1}^n(\partial^2_{\xi_j\xi_k}a(x,\nabla\Phi(x)))(\partial^2_{x_jx_k}\Phi(x))b(x).$$

Note that if Φ solves the ***eikonal equation*** *$a(x,\nabla\Phi(x))\equiv0$, then we have $c_0\equiv0$. The equation $c_1\equiv0$, with b treated as the unknown function, is called the* ***transport equation****. These two equations form the basis of the* ***WKB method*** *of obtaining approximate solutions to the equation $\mathrm{Op}_h(a)u=0$.*

A symplectic geometric interpretation of Exercise 4 is given by

Exercise 5. *(for students familiar with symplectic geometry)*
1. Under the assumptions of the previous exercise, show that

$$\Lambda_\Phi := \{(x, \nabla\Phi(x)) \mid x \in \mathbb{R}^n\}$$

is a Lagrangian submanifold of $\mathbb{R}^{2n}$*, that is the symplectic form* $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$ *vanishes when pulled back to* Λ_Φ*.*

2. Assume a *is real-valued and* $c_0 \equiv 0$*, that is* $a|_{\Lambda_\Phi} = 0$*. Show that the Hamilton vector field* H_a *is tangent to* Λ_Φ*. Next, show that* $c_1 = 0$ *is equivalent (pointwise) to* $(H_a + V)\tilde{b} = 0$ *where*

$$\tilde{b} \in C^\infty(\Lambda_\Phi), \quad \tilde{b}(x, \nabla\Phi(x)) = b(x);$$

$$V \in C^\infty(\Lambda_\Phi; \mathbb{R}), \quad V(x, \nabla\Phi(x)) = \frac{1}{2}\sum_{j,k=1}^n (\partial^2_{\xi_j\xi_k} a(x, \nabla\Phi(x)))(\partial^2_{x_j x_k}\Phi(x)).$$

2.4. **Calculus on manifolds.** We now very briefly discuss semiclassical pseudodifferential operators on manifolds. We consider the case of a compact manifold M. For the noncompact case the theory is similar but the class $S_h(M)$ cannot be invariantly defined so one needs to make some other assumptions (or none) on the growth of symbols as $x \to \infty$. We do not give proofs here; the below results will follow from the pseudodifferential calculus of [Zw, §14.2]. An interested reader can also try to prove the results using Exercise 4.

The class of semiclassical differential operators $\mathrm{Diff}_h^m(M)$ is defined as operators $A : C^\infty(M) \to C^\infty(M)$ with two properties:

(1) A is *local*, that is $\mathrm{supp}(Au) \subset \mathrm{supp}\, u$ for all $u \in C^\infty(M)$;
(2) in any coordinate chart (and cutting off from the boundary of that chart) the operator A corresponds to an element of $\mathrm{Diff}_h^m(\mathbb{R}^n)$.

The principal symbol calculus, Proposition 2.5, still holds, with the following changes:

- The symbols are now functions on the *cotangent bundle*

  $$T^*M = \{(x, \xi) \mid x \in M,\ \xi \in T_x^*M\}$$

  which are polynomials of order m in ξ with coefficients whose derivatives are bounded uniformly in x, h. We denote the space of such polynomials by $\mathrm{Poly}_h^m(T^*M)$. Thus the principal symbol map acts

  $$\sigma_{m,h} : \mathrm{Diff}_h^m(M) \to \frac{\mathrm{Poly}_h^m(T^*M)}{h\,\mathrm{Poly}_h^{m-1}(T^*M)}.$$

- To see why the symbol should be a function on the cotangent bundle (and not, say, tangent bundle), consider the following example:

  $$A = \frac{h}{i}X \in \mathrm{Diff}_h^1(M) \quad \text{where} \quad X \in C^\infty(M; TM) \quad \text{is a vector field.}$$

Then the principal symbol $\sigma_h(A)$ is a linear function on the fibers of T^*M:

$$\sigma_h(A)(x,\xi) = \langle \xi, X(x)\rangle, \quad (x,\xi)\in T^*M$$

where the pairing on the right-hand side makes sense since $X(x)$ is a tangent vector at x and ξ is a cotangent vector at x.

- There is still a quantization procedure $\operatorname{Op}_h : \operatorname{Poly}_h^m(T^*M) \to \operatorname{Diff}_h^m(M)$ (say, defined using a finite covering by charts together with a partition of unity) however there is no *canonical* choice of a quantization procedure. To see this, consider the following quadratic symbol in $\operatorname{Poly}_h^2(T^*M)$:

$$a(x,\xi) = \langle \xi, X(x)\rangle\langle \xi, Y(x)\rangle \quad \text{where} \quad X, Y \in C^\infty(M; TM).$$

  We know that the corresponding operator $\operatorname{Op}_h(a)$ has second order part $-h^2XY$, but what about the lower order terms? In particular, both $-h^2XY$ and $-h^2YX$ are equally good candidates for $\operatorname{Op}_h(a)$ but they differ by the commutator (Lie bracket of vector fields) $h^2[X,Y] \in h\operatorname{Diff}_h^1(M)$.
- The Poisson bracket $\{a,b\}$ can be defined invariantly for $a, b \in C^\infty(T^*M)$. This follows from the fact that T^*M has a natural symplectic form, that is the 2-form

$$\sum_{j=1}^n d\xi_j \wedge dx_j$$

  does not depend on the choice of coordinates.
- To define the formal adjoint A^* we need to fix an inner product on $L^2(M)$ which amounts to fixing a smooth density (volume form) on M. A consequence of the Adjoint Rule is that the choice of this density does not change the leading part of A^*.

References

[Zw] Maciej Zworski, *Semiclassical analysis,* Graduate Studies in Mathematics **138**, AMS, 2012.

E-mail address: dyatlov@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720