MATH 279 HOMEWORK 9

1. Let $P := \frac{1}{2}(-\partial_x^2 + x^2)$ be the one-dimensional (nonsemiclassical) quantum harmonic oscillator and $U(t) := e^{-itP}$. We will look for $U(t)$ in one of the following forms:

$$U(t)u(x) = \frac{a_t}{2\pi} \int_{\mathbb{R}^2} e^{i(\Phi_t(x,\eta) - y\eta)} u(y)\, dy d\eta, \tag{0.1}$$

$$U(t)u(x) = \frac{b_t}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\Psi_t(x,y)} u(y)\, dy. \tag{0.2}$$

Here Φ_t, Ψ_t will be quadratic forms in 2 variables, with coefficients depending smoothly on t in some interval, and a_t, b_t will be functions depending only on t. (This special form of the propagator is due to the particularly nice properties of the harmonic oscillator. The expressions (0.1), (0.2) are oscillatory integrals, known as *metaplectic operators*; they can be viewed as a special case of Fourier integral operators. In your solution you can ignore the issues of their convergence.)

(a) Explain why the case $t = 0$ has the form (0.1) with $\Phi_0(x,\eta) = x\eta$ and $a_0 = 1$.

(b) Writing Φ_t as a generating function of the Hamiltonian e^{tH_p}, $p(x,\xi) = \frac{1}{2}(\xi^2 + x^2)$ (as in the construction of the hyperbolic parametrix), namely

$$(x,\xi) = e^{tH_p}(y,\eta) \iff \xi = \partial_x \Phi_t(x,\eta),\ y = \partial_\eta \Phi_t(x,\eta)$$

and using the Hamilton-Jacobi equation

$$\partial_t \Phi_t(x,\eta) + p(x, \partial_x \Phi_t(x,\eta)) = 0$$

to fix the freedom of adding a function of t only to Φ_t, compute

$$\Phi_t(x,\eta) = -\frac{\tan t}{2} x^2 + \frac{x\eta}{\cos t} - \frac{\tan t}{2} \eta^2. \tag{0.3}$$

This makes sense for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(c) Plug the form (0.1) with the function Φ_t in (0.3) into the evolution equation $(D_t + P)U(t) = 0$ and deduce that $U(t)$ indeed has the form (0.1) for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ where $a_t = (\cos t)^{-1/2}$.

(d) Apply the method of stationary phase in η to (0.1) (here it is exact: the integral just equals the leading term in the expansion. Why?) to see that for $t \in (0, \frac{\pi}{2})$ the operator $U(t)$ also has the form (0.2) with

$$\Psi_t(x,y) = \frac{\cot t}{2} x^2 - \frac{xy}{\sin t} + \frac{\cot t}{2} y^2, \quad b_t = e^{-\frac{i\pi}{4}} (\sin t)^{-1/2}.$$

The above expressions make sense in fact for $t \in (0,\pi)$. Use the evolution equation to show that $U(t)$ is given by the form (0.2) for $t \in (0,\pi)$.

(e) Use (0.2) to compute $U(\frac{\pi}{2})$. From here compute $U(2\pi)$. Can you also compute $U(2\pi)$ using the known spectrum of P?