MATH 279 HOMEWORK 6

In this homework we use the following notation for tensor products:

- if $u, v \in L^2(\mathbb{R}^n)$, we define $u \otimes v \in L^2(\mathbb{R}^{2n})$ by
$$(u \otimes v)(x, y) = u(x)\overline{v(y)};$$
- we also denote by $u \otimes v$ the operator on $L^2(\mathbb{R}^n)$ with the integral kernel $u \otimes v$:
$$(u \otimes v)f = \langle f, v\rangle_{L^2} \cdot u.$$

From the definition of the trace we have
$$\operatorname{tr}(u \otimes v) = \langle u, v\rangle_{L^2}. \tag{0.1}$$

1. Assume that $K(x, y) \in L^2(\mathbb{R}^{2n})$ and consider the integral operator $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by
$$Au(x) = \int_{\mathbb{R}^n} K(x, y)u(y)\, dy. \tag{0.2}$$
Show that A is a Hilbert–Schmidt operator and $\|A\|_2 = \|K\|_{L^2(\mathbb{R}^{2n})}$. (Hint: recall that $\|A\|_2^2 = \sum_{j,k} |\langle Ae_j, f_k\rangle|^2$ for any Hilbert bases $\{e_j\}, \{f_k\}$ of $L^2(\mathbb{R}^n)$. Then use that $\{e_j \otimes f_k\}$ is a Hilbert basis of $L^2(\mathbb{R}^{2n})$.)

2. Assume that $a \in L^2(\mathbb{R}^{2n})$. Show that $a^{\mathrm{w}}(x, hD_x)$ lies in $\mathscr{L}^2(L^2(\mathbb{R}^n))$ and
$$\|a^{\mathrm{w}}(x, hD_x)\|_2 = (2\pi h)^{-n/2}\|a\|_{L^2(\mathbb{R}^{2n})}.$$

3. Recall the quantum harmonic oscillator on $\mathbb{R}^n$
$$P_0 := -h^2\Delta + |x|^2.$$
Let $P_0^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be its inverse.

(a) Using the explicitly known spectrum of P_0, show that for every integer $N > n$ we have $P_0^{-N} \in \mathscr{L}^1(L^2(\mathbb{R}^n))$ and
$$\|P_0^{-N}\|_1 \leq C_N h^{-N}.$$

(b) If $a \in S((1 + |x|^2 + |\xi|^2)^{-n-1})$, show that $a^{\mathrm{w}}(x, hD_x) \in \mathscr{L}^1(L^2(\mathbb{R}^n))$ and
$$\|a^{\mathrm{w}}(x, hD_x)\|_1 \leq C(a)h^{-n-1}$$
where $C(a)$ depends on some $S((1 + |x|^2 + |\xi|^2)^{-n-1})$ seminorm of a.

4. Assume that $K(x,y) \in \mathscr{S}(\mathbb{R}^{2n})$ and let A be the integral operator defined by (0.2). Use the steps below to show that $A \in \mathscr{L}^1(L^2(\mathbb{R}^n))$ and

$$\operatorname{tr} A = \int_{\mathbb{R}^n} K(x,x)\,dx. \tag{0.3}$$

(a) Using the strategy of the previous exercise (showing that $P_0^N A$ is bounded on L^2), show that $A \in \mathscr{L}^1(L^2(\mathbb{R}^n))$ and for N large enough we have

$$\|A\|_1 \le C \sum_{|\alpha|+|\beta|\le N} \sup |z^\alpha \partial_z^\beta K(z)|, \quad z = (x,y). \tag{0.4}$$

Deduce that if $K_n \to K$ in $\mathscr{S}(\mathbb{R}^{2n})$ and (0.3) holds for each K_n, then it also holds for K.

(b) Using (0.1), show that (0.3) holds for $K \in \mathcal{V}$ where $\mathcal{V} \subset \mathscr{S}(\mathbb{R}^{2n})$ consists of linear combinations of functions of the form $K(x,y) = f(x)g(y)$ where $f, g \in \mathscr{S}(\mathbb{R}^n)$.

(c) Show that $\mathcal{V}$ is dense in $\mathscr{S}(\mathbb{R}^n)$. (Hint: if $a \in \mathscr{S}$ then it can be approximated by functions in C_c^∞. Next if $a \in C_c^\infty(\mathbb{R}^{2n})$ then take $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $a = (\chi \otimes \chi)a$ and approximate a by linear combinations of functions of the form $\chi(x)e^{i\langle x,\xi\rangle}\chi(y)e^{i\langle y,\eta\rangle}$.)