MATH 279 HOMEWORK 4

For this homework you might find useful the following formula for the Fourier–Laplace transform of general Gaussian integrals (following by analytic continuation from [Zw, Theorem 3.1]):

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Qw,w\rangle - i\langle w,\zeta\rangle}\,dw = (2\pi)^{n/2} c_Q |\det Q|^{-1/2} e^{-\frac{1}{2}\langle Q^{-1}\zeta,\zeta\rangle} \tag{0.1}$$

where Q is a complex symmetric $n\times n$ matrix, $\operatorname{Re} Q$ is positive definite, $\zeta\in\mathbb{C}^n$, $\langle z,w\rangle := \sum_j z_j w_j$, and c_Q is a constant depending on Q such that $|c_Q| = 1$.

1. Fix $\delta \geq 0$ and consider the symbol on $\mathbb{R}^{2n}$

$$a(x,\xi;h) := a_0\Big(\frac{x}{h^\delta},\frac{\xi}{h^\delta}\Big) \quad \text{where} \quad a_0(x,\xi) := \exp\Big(-\frac{|x|^2+|\xi|^2}{2}\Big).$$

(a) Show that $a \in S_\delta(1)$.

(b) When $\delta \leq \frac{1}{2}$ show that

$$\|a^{\mathrm{w}}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \geq c > 0$$

for some h-independent constant c. (Hint: apply the operator to $u(x;h) = \exp(-\frac{|x|^2}{2h^{2\delta}})$, computing $a^{\mathrm{w}}u$ using (0.1).)

(c) When $\delta > \frac{1}{2}$ show that

$$\|a^{\mathrm{w}}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C h^{n(\delta-\frac{1}{2})}$$

for some h-independent constant C. (Hint: bound the operator norm by the L^2 norm of the integral kernel.)

2. Let a be as above. Let $a\#a$ be defined in [Zw, Theorem 4.11].

(a) Using (0.1) show that

$$a\#a(x,\xi) = \Big(1+\frac{1}{4}h^{2-4\delta}\Big)^{-n} \exp\Big(-\frac{|x|^2+|\xi|^2}{h^{2\delta}+\frac{1}{4}h^{2-2\delta}}\Big).$$

(b) For which values of δ can we say that $a\#a = a^2 + o(1)$ (uniformly on compact sets in (x,ξ)) as $h\to 0$?

References

[Zw] Maciej Zworski, *Semiclassical analysis,* Graduate Studies in Mathematics **138**, AMS, 2012.