Some trig identities for Math 1B

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1. The fundamental identity:

$$\cos^2\theta + \sin^2\theta = 1 \tag{1}$$

Dividing this identity by $\cos^2\theta$ or $\sin^2\theta$, we get

$$\cot^2\theta + 1 = \csc^2\theta, \;\; 1 + \tan^2\theta = \sec^2\theta. \tag{2}$$

2. Sums, differences, and products:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \tag{3}$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B, \tag{4}$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B, \tag{5}$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B. \tag{6}$$

Note that (4) can be obtained from (3) and (6) can be obtained from (5) by substituting $-B$ instead of B and using the identities

$$\cos(-x) = \cos x, \;\; \sin(-x) = -\sin x. \tag{7}$$

By adding (3) and (4), adding (5) and (6), and subtracting (3) from (4), we get

$$\cos A\cos B = (\cos(A+B) + \cos(A-B))/2, \tag{8}$$
$$\sin A\cos B = (\sin(A+B) + \sin(A-B))/2, \tag{9}$$
$$\sin A\sin B = (\cos(A-B) - \cos(A+B))/2. \tag{10}$$

When $A = B$, (3) and (5) become

$$\cos 2A = \cos^2 A - \sin^2 A, \tag{11}$$
$$\sin 2A = 2\sin A\cos A. \tag{12}$$

Using (1), we then get

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A, \tag{13}$$
$$\cos^2 A = \frac{1+\cos 2A}{2}, \;\; \sin^2 A = \frac{1-\cos 2A}{2}. \tag{14}$$

3. Differentiation:

$$d(\cos x) = -\sin x\,dx, \;\; d(\sin x) = \cos x\,dx, \tag{15}$$

$$d(\tan x) = \sec^2 x\,dx, \;\; d(\cot x) = -\csc^2 x\,dx, \tag{16}$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C, \;\; \int \csc x\,dx = \ln|\csc x - \cot x| + C. \tag{17}$$

4. Integrating trigonometric expressions:

   - If our expression contains the factor $\sin x\,dx$ and the remaining part is a function of $\cos x$, $\sin^2 x$, $\tan^2 x$, and $\cot^2 x$, then do the substitution $u = \cos x$ and then use the identities (1)–(2) to express everything as a function of u. Example:

     $$\int \ln(\cos x)\sin^3 x\,dx = \int \ln(\cos x)\sin^2 x(\sin x\,dx)$$
     $$= \int (1-\cos^2 x)\ln(\cos x)d(-\cos x) = -\int (1-u^2)\ln u\,du.$$

     Similarly, if our expression contains the factor $\cos x\,dx$ and the remaining part is a function of $\sin x$, $\cos^2 x$, $\tan^2 x$, and $\cot^2 x$, then do the substitution $u = \sin x$.

   - If our expression contains only squares of trigonometric functions ($\cos^2 x, \sin^2 x$, etc.) and the product $\cos x \sin x$, then use the identities (14) and (12) to get an expression in terms of $\cos 2x$ and $\sin 2x$ and then do the substitution $u = 2x$. Example:

     $$\int \frac{\cos^2 x}{1 + 2\sin x \cos x}\,dx = \frac{1}{2}\int \frac{1+\cos 2x}{1+\sin 2x}\,dx$$
     $$= \frac{1}{4}\int \frac{1+\cos u}{1+\sin u}\,du.$$

   - If our expression contains the factor $\sec^2 x\,dx$ and the remaining part is a function of $\tan x$, $\cot x$, $\cos^2 x$, and $\sin^2 x$, then do the substitution $u = \tan x$ and use the identities (2) to express everything as a function of u. Example:

     $$\int \sec^4 x\,dx = \int \sec^2 x(\sec^2 x\,dx)$$
     $$= \int (1+\tan^2 x)\,d(\tan x) = \int 1 + u^2\,du.$$

     Similarly, one can use the substitution $u = \cot x$ in case we have the factor $\csc^2 x\,dx$.