Math 1B worksheet

Sep 21, 2009

Please split into groups of 2–4 people and solve the problems on the board. Please write the solutions as clearly as possible. You may pick the order in which to do the problems, but do not attempt the problems marked with * until you are sure that you have done (or can do easily) everything else. You may use the book to look up the necessary formulas.

1. Find the length of the curve given by the equation $y = \cosh x$ for $-1 \leqslant x \leqslant 1$.

2. Set up an integral to compute the surface obtained by rotating the curve $y^2 - x^2 = 1$, $0 \leqslant x \leqslant 1$, $y > 0$, around the x axis. What if we rotate around the y axis instead?

3. Find the centroid of the shape bounded by the lines $y = \sin x$ and $y = 2\sin x$ for $0 \leqslant x \leqslant \pi$.

4. Find the hydrostatic force of a liquid of density ρ against one side of a vertical disc of radius 2 whose center lies at depth 1.

5. Find the area of the surface obtained by rotating the curve $y = e^x$, $0 \leqslant x \leqslant \ln 2$, around the x-axis.

100* Let $f(x)$, $a \leqslant x \leqslant b$, be an increasing function and let $g(y)$, $f(a) \leqslant y \leqslant f(b)$ be its inverse function. Prove that the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is the same as the length of the curve $x = g(y)$, $f(a) \leqslant y \leqslant f(b)$. Give a geometric explanation to this fact.

101* Find a formula for a centroid of the curve $y = f(x)$, $a \leqslant x \leqslant b$, with mass distributed uniformly along the curve's length.

Hints and answers

1. Use that the derivative of cosh is sinh and that $\cosh x^2 = \sinh x^2 + 1$.

Answer: $e - e^{-1}$.

2. We can write $y = \sqrt{x^2+1}$ or $x = \sqrt{y^2-1}$. When solving the second part, do not forget to put the right limits of integration!

Answer: $2\pi \int_0^1 \sqrt{2x^2+1}\, dx$; $2\pi \int_1^{\sqrt{2}} \sqrt{2y^2-1}\, dy$.

3. The shape is symmetric about the line $x = \pi/2$, so the x-coordinate of the centroid is $\pi/2$. For the y coordinate, do not forget that moments, not centroids, are additive!

Answer: $(\pi/2, 3\pi/8)$.

4. We need to compute the integral $2\rho g \int_{-2}^{1} (1-y)\sqrt{4-y^2}\, dy$. For $\int_{-2}^{1} \sqrt{4-y^2}\, dy$, use a trigonometric substitution (say, $y = 2\sin\theta$, where $-\pi/2 \leqslant \theta \leqslant \pi/6$). For $\int_{-2}^{1} y\sqrt{4-y^2}\, dy$, use the substitution $u = 4 - y^2$ (if you do the substitution in the definite integral, note that the function $f(y) = 4 - y^2$ is not one-to-one, so we have to split our integral into $\int_{-2}^{0}$ and $\int_0^1$).

Answer: $2\rho g\left(\frac{4\pi}{3} + \frac{3\sqrt{3}}{2}\right)$.

5. We need to compute the integral $2\pi \int_0^{\ln 2} e^x \sqrt{1+e^{2x}}\, dx$. After the substitution $y = e^x$, we get the integral $2\pi \int_1^2 \sqrt{1+y^2}\, dy$. One way to compute this integral is to make the substitution $y = \sinh t$; we then get the integral $2\pi \int \cosh^2 t\, dt$, which can be evaluated once we write $\cosh t = \frac{e^t + e^{-t}}{2}$. To find how t depends on y, we may use that the equation $y = \sinh t = \frac{e^t + e^{-t}}{2}$ can be multiplied by e^t to obtain a quadratic equation in e^t. Another way to compute the integral $\int \sqrt{1+y^2}\, dy$ is to make a trigonometric substitution $y = \tan\theta$; then the integral becomes $\int \sec^3\theta\, d\theta$. One then can either use Example 7.2.8 in Stewart or make another substitution $u = \sin\theta$ to get the integral $\int \frac{du}{(1-u^2)^2}$, which can be computed using partial fractions. Both ways will give after simplification $\int \sqrt{1+y^2}\, dy = \frac{1}{2}(y\sqrt{y^2+1} + \ln(y + \sqrt{y^2+1})) + C$.

Answer: $\pi(2\sqrt{5} + \ln(2+\sqrt{5}) - \sqrt{2} - \ln(1+\sqrt{2}))$.

100. Use the substitution $y = f(x)$ and the formula for the derivative of an inverse function.

101. One way to do this problem is to employ Riemann sums similarly to the textbook.

Answer: $x_c = \frac{1}{L}\int_a^b x\, ds$, $y_c = \frac{1}{L}\int_a^b f(x)\, ds$, where L is the length of the curve and $ds = \sqrt{1 + (f'(x))^2}\, dx$.