Math 1B worksheet

Sep 14, 2009

Please split into groups of 3–4 people and solve the problems on the board. Please write the solutions as clearly as possible. You may pick the order in which to do the problems, but do not attempt the last two problems until you are sure that you have done (or can do easily) everything else. It is suggested that you start by solving one of the first two problems, then one of the problems 3–4, then one of the problems 5–6. You may use your calculators to do arithmetics (say, add numbers), but you may not use them to compute the integrals.

1. Calculate $\int_0^{\pi} \sin x \, dx$ using Simpson's Rule with $n = 4$. Compare the answer with the exact value of the integral.

2. Calculate $\int_1^3 \frac{dx}{x}$ using left endpoint approximation with $n = 4$. Deduce from the graph whether your answer is larger or smaller than the exact value of the integral.

3. Does the integral $\int_1^{\infty} x e^{-x} \, dx$ converge? If so, evaluate it.

4. Does the integral $\int_{-1}^{1} \frac{e^{|x|}}{\sin x} \, dx$ converge? If so, evaluate it.

5. Compute the indefinite integral $\int \frac{e^{2t}}{1 + e^{4t}} \, dt$.

6. Compute the indefinite integral $\int \arctan \sqrt{x} \, dx$.

7* For which values of a and b does the integral $\int_2^{\infty} \frac{x^a}{(\ln x)^b} \, dx$ converge?

8* Compute the integral $\int_{-\infty}^{\infty} e^{-x^2} \, dx$ with error no more than 0.1.

Hints and answers

1. The answer is $\frac{\pi}{6}(1+2\sqrt{2}) \approx 2.004$ and the exact value of the integral is 2.

2. The answer is $\frac{77}{60} \approx 1.283$ and the exact value of the integral is $\ln 3 \approx 1.0986$. It is smaller than the answer because the function $\frac{1}{x}$ is decreasing.

3. Using integration by parts, we get $\int xe^{-x}\,dx = -(x+1)e^{-x}+C$. The function $F(x) = -(x+1)e^{-x}$ is continuous on the interval $[1,\infty)$ and $\lim_{x\to+\infty} F(x) = 0$. Therefore, the integral converges and is equal to $-F(1) = 2e^{-1}$.

4. The function under the integral sign is odd, so it is enough to study the integral $\int_0^1 \frac{e^{|x|}}{\sin x}\,dx$. Now, the function $f(x) = e^{|x|}/\sin x$ is continuous on $(0,1]$ and we have $\lim_{x\to 0} \frac{xe^{|x|}}{\sin x} = 1$, so there exist $\delta > 0$ and a constant $C > 0$ such that $0 < C|x|^{-1} < \frac{e^{|x|}}{\sin x}$ for $|x| < \delta$. However, the integral $\int_0^1 \frac{dx}{x}$ is divergent; therefore, our integral is divergent by the Comparison Theorem.

Alternatively, note that $e^{|x|} \geqslant 1$ and thus for $x > 0$, $\frac{e^{|x|}}{\sin x} \geqslant \csc x$; we may compute $\int \csc x\,dx = \ln \frac{1-\cos x}{\sin x} + C = F(x) + C$. Now, $\lim_{x\to 0} \frac{1-\cos x}{\sin x} = 0$ (say, by L'Hôpital's rule), so $\lim_{x\to 0} F(x) = \lim_{t\to 0} \ln t = -\infty$; the integral $\int_0^1 \csc x\,dx$ diverges, so our integral also diverges.

5. Make the substitution $u = e^{2t}$.

Answer: $\frac{1}{2}\arctan e^{2t} + C$.

6. Make the substitution $t = \sqrt{x}$ to get the integral $2\int t \arctan t\,dt$. Now, integrate by parts with $u = \arctan t$ and $dv = 2t dt$: $du = \frac{dt}{1+t^2}$, $v = t^2$.

Answer: $(x+1)\arctan\sqrt{x} - \sqrt{x} + C$.

7. If $a \neq 1$, use the Comparison Theorem together with the inequalities $0 < c < \ln x < Cx^{\delta}$ for any δ and some constants c and C (the latter depending on δ). If $a = 1$, make the substitution $u = \ln x$.

Answer: The integral converges for $a < -1$ and also for $a = -1$ and $b > 1$.

8. Note that the function e^{-x^2} is even, so we only need to compute $\int_0^\infty e^{-x^2}\,dx$. Then find some A for which $\int_A^\infty e^{-x^2}\,dx < \frac{1}{40}$, estimating e^{-x^2} by a function with known antiderivative (say, xe^{-x^2} for $x > 1$). Finally, compute $\int_0^A e^{-x^2}\,dx$ using your favourite approximate integration method with error no more than $\frac{1}{20}$. This integral can be computed explicitly (using multivariable calculus or complex analysis) and its exact value is $\sqrt{\pi} \approx 1.772$.