Math 1B worksheet

Oct 7, 2009

1–2. Determine whether we may apply the Alternating Series Test to conclude that the following series converge. If so, estimate the error $|s - s_n|$, where s in the sum of the series and s_n is the sum of the first n terms:

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}, \tag{1}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{10^n}. \tag{2}$$

3–4. Use the Ratio Test to find whether the following series are convergent, divergent, or the test is inconclusive:

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}, \tag{3}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n^2 2^n}{n!}. \tag{4}$$

5–6. Use the Root Test to find whether the following series are convergent, divergent, or the test is inconclusive:

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{e^{n^2}}{n^n}. \tag{6}$$

7–12. Determine whether the following series converges absolutely, converges conditionally, or diverges: (For problem 7, find, how many terms of the series we have to take to compute the sum with error no more than 0.01. In problem 9, find for which real values of k the series converges absolutely, for which values it converges conditionally, and for which k it diverges)

$$\sum_{n=1}^{\infty} \left(\frac{\sin n}{n} \right)^3, \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{1 + (-1)^n \cdot n}{n^2}, \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^k}, \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{10^n (n!)^2}{(2n)!}, \tag{10}$$

$$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}, \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}. \tag{12}$$

Hints and answers

1. We have $\frac{\cos(n\pi)}{n^{3/4}} = (-1)^n b_n$, where $b_n = \frac{1}{n^{3/4}}$ is monotonely decreasing and goes to zero as $n \to \infty$.

Answer: Yes; $|s - s_n| \leqslant \frac{1}{(n+1)^{3/4}}$.

2. Put $b_n = \frac{n}{10^n}$. Then $b_n \to 0$ as $n \to \infty$. It remains to verify that b_n is decreasing; for that, it is enough to prove that $\frac{b_{n+1}}{b_n} \leqslant 1$. However, the latter is $\frac{1}{10}\left(1 + \frac{1}{n}\right) \leqslant 1$ for $n \geqslant 1$.

Answer: Yes; $|s - s_n| \leqslant \frac{n+1}{10^{n+1}}$.

3. Put $a_n = \frac{n^2}{2^n}$ and compute $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2$. The limit of this is $\frac{1}{2}$.

Answer: Converges.

4. Put $a_n = (-1)^{n+1}\frac{n^2 2^n}{n!}$ and compute $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1}\left(1 + \frac{1}{n}\right)^2$. The limit of this is 0.

Answer: Converges.

5. Put $a_n = \frac{(-2)^n}{n^n}$ and compute $\sqrt[n]{|a_n|} = \frac{2}{n}$. The limit of this is 0.

Answer: Converges.

6. Put $a_n = \frac{e^{n^2}}{n^n}$ and compute $\sqrt[n]{|a_n|} = \frac{e^n}{n}$. The limit of this is ∞.

Answer: Diverges.

7. Put $a_n = \left(\frac{\sin n}{n}\right)^3$; then $|a_n| \leqslant \frac{1}{n^3}$. By Comparison Test, the series $\sum_{n=1}^{\infty} |a_n|$ converges; therefore, our series converges absolutely. The error $|s - s_n|$ can be estimated by $\sum_{m=n+1}^{\infty} \frac{1}{m^3}$; by error estimates for the integral test, $|s - s_n| \leqslant \int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2}$. We then need to choose n for which $\frac{1}{2n^2} \leqslant 0.01$.

Answer: Converges absolutely; $n = 8$ would be enough.

8. Write the series as the sum of the absolutely convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$; the latter converges conditionally by p-series test and alternating series test.

Answer: Converges conditionally.

9. Put $a_n = (-1)^n \frac{e^{1/n}}{n^k}$; then $|a_n| = \frac{e^{1/n}}{n^k}$. If $k < 0$, then $\lim_{n\to\infty} \frac{1}{n^k} = \infty$ and $\lim_{n\to\infty} a_n$ does not exist; the series diverges by Test for Divergence. If $k = 0$, then $\frac{1}{n^k} = 1$ and $\lim_{n\to\infty} e^{1/n} = 1$, so $\lim_{n\to\infty} a_n$ does not exist and the series diverges. Now, let $k > 0$. For absolute convergence, note that $\lim_{n\to\infty} \frac{|a_n|}{1/n^k} = 1$; therefore, by Limit Comparison Test $\sum_{n=1}^{\infty} |a_n|$ converges for $k > 1$. Finally, for $0 < k \leqslant 1$ the series converges conditionally by Alternating Series Test.

Answer: Converges absolutely for $k > 1$, converges conditionally for $0 < k \leqslant 1$, and diverges for $k \leqslant 0$.

10. Let $a_n = \frac{(10)^n (n!)^2}{(2n)!}$ and apply the Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{10(n+1)^2}{(2n+1)(2n+2)} \to 2.5$ as $n \to \infty$.

Answer: Diverges.

11. Let $a_n = \frac{n}{(\ln n)^n}$ and apply the Root Test: $\sqrt[n]{|a_n|} = \frac{\sqrt[n]{n}}{\ln n} \to 0$ as $n \to \infty$ because $\ln n \to \infty$ and $\sqrt[n]{n} = e^{\frac{\ln n}{n}} \to 1$.

Answer: Converges absolutely.

12. Let $a_n = \frac{n!}{n^n}$ and apply the Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left(1 + \frac{1}{n}\right)^{-n}$ converges to e^{-1}.

Answer: Converges.