Math 1B worksheet

Oct 5, 2009

1–9. Determine whether the following series converge or diverge. (In problem 3, determine, for which integer values of k the series converges.)

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2+1}, \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+1}, \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{2+3\cdot n^k}{3+2\cdot n^{2k}}, \tag{3}$$

$$\sum_{n=1}^{\infty} \sin(1/n^2), \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}, \tag{6}$$

$$\sum_{n=1}^{\infty} \sqrt{1-\cos(1/n)}, \tag{7}$$

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}, \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n!}. \tag{9}$$

Hints and answers

We write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and lies strictly between 0 and ∞ (in other words, if we may apply the limit comparison test).

1. We have $\frac{\sin^2 n}{n^2+1} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Answer: Converges.

2. We have $\frac{\sqrt{n+1}}{n^2+1} \sim \frac{1}{n\sqrt{n}}$.

Answer: Converges.

3. Let $a_n = \frac{2+3\cdot n^k}{3+2\cdot n^{2k}}$. For $k < 0$, we have $\lim_{n\to\infty} a_n = \frac{2}{3}$, so the series diverges by Test for Divergence. For $k = 0$, we have $a_n = 1$, so the series also diverges. For $k > 0$, we have $a_n \sim \frac{1}{n^k}$.

Answer: Converges for $k > 1$, diverges for all other k.

4. We have $\sin(1/n^2) \sim \frac{1}{n^2}$ since $\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Answer: Converges.

5. We write $n+4^n = 4^n(1+\frac{n}{4^n})$ and $\lim_{n\to\infty} \frac{n}{4^n} = 0$; therefore, $n+4^n \sim 4^n$. Similarly, $n+6^n \sim 6^n$, so our series is equivalent to $\sum_{n=1}^{\infty} \frac{4^n}{6^n}$, a geometric series.

Answer: Converges.

6. Divide and multiply by $\sqrt{n+1} - \sqrt{n}$ to get that our series is equivalent to $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$.

Answer: Converges.

7. We have $1 - \cos(1/n) = 2\sin^2(1/(2n))$ by double angle formula, so $\sqrt{1-\cos(1/n)} \sim \sin(1/(2n)) \sim 1/n$.

Answer: Diverges.

8. We have $\lim_{n\to\infty}(1+1/n)^2 = 1$, so our series is equivalent to the geometric series $\sum_{n=1}^{\infty} e^{-n}$.

Answer: Converges.

9. We write $n! = 1 \cdot 2 \cdots n$ and replace all the terms except the first one by 2, to get $n! \geq 2^{n-1}$. So, $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$; the latter is a geometric series. (Many more ways to compare exist; for example, $\frac{1}{n!} \leq \frac{1}{n(n-1)}$ for $n \geq 2$.)

Answer: Converges.