Math 1B, Section 105 Your name: Key
Quiz 9, November 18, 2009

Please write your name on each sheet. Show your work clearly and in order, including the intermediate steps in the solutions and the final answer.

1. (7 pt) Find the general solution for the differential equation

$$y' \cos x + y \sin x = 1, \quad -\pi/2 < x < \pi/2.$$

Find the solution satisfying $y(0) = 1$.

Divide by $\cos x$: $y' + \tan x \cdot y = 1/\cos x$.

Integrating factor: $I = e^{+\int \tan x \, dx} = e^{-\ln \cos x} = \dfrac{1}{\cos x}$.

General solution: $y = \dfrac{1}{I}\left(\int \dfrac{I \, dx}{\cos x} \right) = \cos x \cdot \int \dfrac{dx}{\cos^2 x} =$

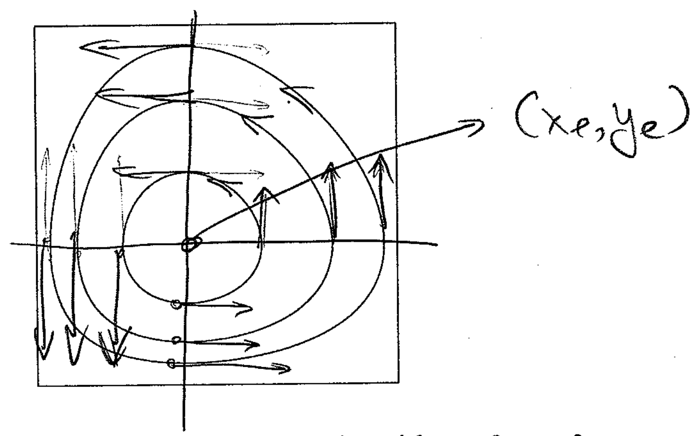
$= \cos x (\tan x + C) = \boxed{\sin x + C \cos x}$

$y(0) = 1 \Rightarrow 0 + C \cdot 1 = 1 \Rightarrow C = 1 \Rightarrow \boxed{y(x) = \sin x + \cos x}$

2. (6 pt) Consider the differential equation

$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = -2y + xy.$$

Here is the graph of several of its trajectories:



(a) Find the coordinates of the equilibrium point with $x > 0$, $y > 0$.

(b) Explain how to find the equilibrium point on the graph.

(c) As t increases, does the point $(x(t), y(t))$ go clockwise or counterclockwise along the given trajectories? Explain.

(a) Solve the system $\begin{cases} x_e - x_e y_e = 0 \\ -2y_e + x_e y_e = 0 \end{cases} \rightarrow \begin{matrix} y_e = 1 \\ x_e = 2 \end{matrix} \Rightarrow \boxed{(2,1)}$

(b) For $x = x_e$, we have $dy/dt = 0$, so the trajectory should have a horizontal tangent vector.
For $y = y_e$, the trajectory has a vertical tangent vector

(c) If $y = y_e$, $x > x_e$, then $\frac{dy}{dt} > 0 \Rightarrow$ the tangent vector points up. So, (x,y) goes $\boxed{\text{counterclockwise}}$

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3. (7 pt) Consider the logistic equation

$$y'(x) = y(1-y).$$

(a) Find the equilibrium points and draw the corresponding solutions in the (x, y) plane.

(b) Find the general solution of the equation (don't forget the solutions corresponding to the equilibrium points).

(c) Find the two solutions satisfying $y(0) = 1/2$ and $y(0) = 2$. For each of them, find the maximal interval containing $x = 0$ on which they are defined and the limits of $y(x)$ as x approaches each endpoint of the interval. Using this information, sketch the graphs of the two solutions on the same axes as the graphs from (a).

(a) We solve the equation $y(1-y)=0$: $\boxed{y=0,1}$ are the equilibrium points.

(b) If $y(1-y)\neq 0$, then we can divide by it:

$$\frac{dy}{dx} = y(1-y) \Rightarrow \frac{dy}{y(1-y)} = dx \Rightarrow \int \frac{dy}{y(1-y)} = \int dx \Rightarrow$$

$$\Rightarrow \int \frac{dy}{y} + \int \frac{dy}{1-y} = x + C \Rightarrow \ln|y| - \ln|1-y| = x + C \Rightarrow$$

$$\Rightarrow \ln\left|\frac{y}{1-y}\right| = x + C \Rightarrow \frac{y}{1-y} = \tilde{C}e^x, \text{ where } \tilde{C}\neq 0 \Rightarrow$$

$$\Rightarrow y = \tilde{C}e^x - \tilde{C}e^x \cdot y \Rightarrow y = \frac{\tilde{C}e^x}{1+\tilde{C}e^x}.$$

The general solution is:

$$\boxed{y = \frac{\tilde{C}e^x}{1+\tilde{C}e^x}, \tilde{C}\neq 0; \ y=0; \ y=1}$$

(c) If $y(0) = \frac{1}{2}$ and $y = \frac{\tilde{C}e^x}{1+\tilde{C}e^x}$, then $\frac{1}{2} = \frac{\tilde{C}}{1+\tilde{C}} \Rightarrow \tilde{C} = 1.$

$$\boxed{y = \frac{e^x}{1+e^x}}$$

- Defined for all x
- $\lim_{x\to+\infty} y = 1$
- $\lim_{x\to-\infty} y = 0$

If $y(0)=2$, then $2 = \frac{\tilde{C}}{1+\tilde{C}} \Rightarrow \tilde{C} = -2;$

$$\boxed{y = \frac{-2e^x}{1-2e^x}}$$

- Defined for $1-2e^x<0$; so, for $x > -\ln 2$
- $\lim_{x\to+\infty} y = 1$
- $\lim_{x\to-\ln 2+0} y = \frac{-1}{\lim_{x\to-\ln 2+0} 1-2e^x} = +\infty$

Problem 3, continued: