Math 1B practice midterm

Nov 1, 2009

1. Do the following series converge absolutely, converge conditionally, or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[7]{n+1} - \sqrt[7]{n}}{n}$ (Hint: carry $\sqrt[7]{n}$ out of the numerator and use Taylor series to analyse what remains.)

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(n+5)\ln n}$.

2. Assume that the series $\sum_{n=1}^{\infty} c_n x^n$ converges for $x = 1$. Does it always converge for $x = -1$? What about $x = 1/2$ or $x = 2$? Can you say anything about the limit $\lim_{n\to\infty} c_n$? (For each statement, either prove it or give a counterexample.)

3. Write a series for $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$. Using the series, describe how one can compute the given integral with an error no more than 0.001.

4. Find the radius and the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Find $f^{(10)}(0)$.

5. Find the Maclaurin series for the function $f(x) = \sqrt{1 - \cos x}$.

6. (a) Solve the initial value problem

$$\begin{cases} y' = 2xy^2, \\ y(0) = 1/2. \end{cases}$$

(b) Use Euler's method with step 1/3 to find $y(1)$ approximately. What is the exact value of $y(1)$?

Hints and answers

1. (a) We have

$$\frac{\sqrt[7]{n+1}-\sqrt[7]{n}}{n} \sim n^{-6/7}\left(\left(1+\frac{1}{n}\right)^{1/7}-1\right) \sim \frac{1}{7}n^{-13/7}.$$

Since $13/7 > 1$, the series converges absolutely. (Note that all the terms are positive.)

(b) The series does not converge absolutely because $\frac{1}{(n+5)\ln n} \sim \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by Integral Test. We then use the Alternating Series Test to conclude that the series converges conditionally.

2. Answer: Has to converge at $x = 1/2$, does not have to converge at $x = -1$ or $x = 2$. A counterexample for the latter two would be $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$.

3. We have $\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$; therefore, our integral is

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}.$$

The series above converges by Alternating Series Test; therefore, we may use the error estimate to get

$$\left| I - \sum_{n=0}^{m-1} \frac{(-1)^n}{(2n+1)(2n+1)!} \right| \leqslant \frac{1}{(2m+1)(2m+1)!}.$$

So, we may take the sum of the first m terms (from $n = 0$ to $n = m - 1$) as long as $(2m+1)(2m+1)! \geqslant 1000$; this is true starting from $m = 3$.

4. $R = 1$, interval of convergence is $[-1, 1]$, $f^{(10)}(0) = 10!/100 = 9!/10 = 36288$. (Leaving the answer as $10!/10$ is okay.)

5. For $-\pi/2 \leqslant x \leqslant \pi/2$, we have $f(x) = \sqrt{2}\sin(x/2)$; using Maclaurin series for $\sin x$, we get the

Answer: $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n-1/2}}{(2n+1)!} x^{2n+1}$.

6. (a) Using separation of variables, we get the family of solutions $y = \frac{1}{C-x^2}$. Using the initial condition, we get $C = 2$.