Math 1B, Section 106 Your name: Key

Quiz 10, December 2, 2009

Please write your name on each sheet. Show your work clearly and in order, including the intermediate steps in the solutions and the final answer.

1. (7 pt) Find the general solution of the equation

$$y'' - y = e^{-x} + x \sin x.$$

Find the solution satisfying $y(0) = 0$, $y'(0) = 1$.

Homogeneous equation: $y'' - y = 0$

Auxiliary equation: $r^2 - 1 = 0$, roots $r = \pm 1$,

General soln of the hom. equ: $y = C_1 e^x + C_2 e^{-x}$

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Let us find $y_1$ with $y_1'' - y_1 = e^{-x}$.

Since $e^{-x}$ solves the hom. equ, we look for $y_1$ in the form $y_1 = Axe^{-x}$; $y_1' = A(-xe^{-x} + e^{-x}) = A(1-x)e^{-x}$, $y_1'' = A(-e^{-x} - (1-x)e^{-x}) = A(x-2)e^{-x}$.

So, $y_1'' - y_1 = -2Ae^{-x} = e^{-x} \Rightarrow A = -\frac{1}{2} \Rightarrow y_1 = -\frac{1}{2}xe^{-x}$

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Now, let us find $y_2$ with $y_2'' - y_2 = x \sin x$. We look for it in the form $y_2 = (Bx + C)\sin x + (Dx + E)\cos x$;

then $y_2' = (Bx+C)\cos x + B\sin x - (Dx+E)\sin x + D\cos x =$

$= (Bx + C + D)\cos x + (-Dx - E + B)\sin x$;

$y_2'' = -(Bx + C + D)\sin x + B\cos x + (-Dx - E + B)\cos x -$

$- D\sin x = -(Bx + C + 2D)\sin x + (-Dx - E + 2B)\cos x$;

$y_2'' - y_2 = -2(Bx + C + D)\sin x - 2(Dx + E - B)\cos x = x\sin x$.

So, $\underset{(x\sin x)}{-2B = 1}$, $\underset{(\sin x)}{C + D = 0}$, $\underset{(x\cos x)}{D = 0}$, $\underset{(\cos x)}{E - B = 0}$.

Therefore, $B = -\frac{1}{2}$, $E = -\frac{1}{2}$, $C = D = 0 \Rightarrow y_2 = -\frac{1}{2}x\sin x - \frac{1}{2}\cos x$.

2. (3 pt) Consider the equation

$$4x''(t) + cx'(t) + 9x(t) = 0.$$

(a) For which positive values of c do we have underdamping, critical damping, or overdamping? Explain.

(b) Take your favorite value of c for which we have underdamping and find two linearly independent solutions to the differential equation.

$144 = 12^2$

(a) Auxiliary equation: $4r^2 + cr + 9 = 0$; roots: $r = \frac{-c \pm \sqrt{c^2 - 144}}{8}$.

$c > 12$: $c^2 > 144$, 2 real roots $\rightarrow$ overdamping

$c = 12$: $c^2 = 144$, 1 real root $\rightarrow$ critical damping

$c < 12$: $c^2 < 144$, no real roots $\rightarrow$ underdamping

(b) Take $c = \sqrt{143}$: $r = \frac{-\sqrt{143} \pm \sqrt{-1}}{8} = -\frac{\sqrt{143}}{8} \pm \frac{i}{8}$.

$x_1(t) = e^{-\frac{\sqrt{143}}{8}t} \cos(t/8)$, $x_2(t) = e^{-\frac{\sqrt{143}}{8}t} \sin(t/8)$

(problem 1, continued)

General solution: $\boxed{y = -\frac{1}{2} x e^{-x} - \frac{1}{2} x \sin x - \frac{1}{2}\cos x + c_1 e^{x} + c_2 e^{-x}}$

Initial conditions: $0 = y(0) = -\frac{1}{2} + c_1 + c_2 \Rightarrow c_1 + c_2 = \frac{1}{2}$

$1 = y'(0) = -\frac{1}{2} + c_1 - c_2 \Rightarrow c_1 - c_2 = \frac{3}{2}$

$c_1 = \frac{1}{2}\left(\frac{1}{2} + \frac{3}{2}\right) = 1$, $c_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{3}{2}\right) = -\frac{1}{2}$

Answer: $\boxed{y = -\frac{1}{2} x e^{-x} - \frac{1}{2} x \sin x - \frac{1}{2}\cos x + e^{x} - \frac{1}{2} e^{-x}}$