Math 1B, Section 105 Your name: Key

Quiz 10, December 2, 2009

Please write your name on each sheet. Show your work clearly and in order, including the intermediate steps in the solutions and the final answer.

1. (7 pt) Find the general solution of the equation

$$y'' - y = e^{-x} + x\cos x.$$

Find the solution satisfying $y(0) = 1$, $y'(0) = 0$.

Auxiliary equation: $r^2 - 1 = 0 \to$ roots $r = \pm 1$

General solution of the homogeneous eqn: $C_1 e^x + C_2 e^{-x}$

$y_1'' - y_1 = e^{-x} \to$ look for $y_1 = Axe^{-x}$ since e^{-x} solves the homogeneous eqn. Then, $y_1'' - y_1 = A((xe^{-x})'' - (xe^{-x})) =$

$= A(((1-x)e^{-x})' - xe^{-x}) = A((x-2)e^{-x} - xe^{-x}) = -2Ae^{-x} = e^{-x}$.

So, $A = -\frac{1}{2} \Rightarrow y_1 = -\frac{1}{2}xe^{-x}$.

$y_2'' - y_2 = x\cos x \to$ look for $y_2 = (Bx+C)\cos x + (Dx+E)\sin x$. Then

$y_2' = (-Bx - C + D)\sin x + (Dx + E + B)\cos x$;

$y_2'' = (-Bx - C + 2D)\cos x - (Dx + E + 2B)\sin x$;

$y_2'' - y_2 = -2(Bx + C - D)\cos x - 2(Dx + E + B)\sin x = x\cos x$.

We get:
$$\begin{cases} -2B = 1 \\ C - D = 0 \\ D = 0 \\ E + B = 0 \end{cases} \quad \begin{matrix} (x\cos x) \\ (\cos x) \\ (x\sin x) \\ (\sin x) \end{matrix} \quad \to \quad \begin{matrix} B = -\frac{1}{2}, \\ C = D = 0, \\ E = \frac{1}{2}. \end{matrix}$$

$y_2 = -\frac{1}{2}x\cos x + \frac{1}{2}\sin x$.

General solution: $y = -\frac{1}{2}xe^{-x} - \frac{1}{2}x\cos x + \frac{1}{2}\sin x + C_1 e^x + C_2 e^{-x}$.

$y' = \frac{1}{2}(x-1)e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}x\sin x + \frac{1}{2}\cos x + C_1 e^x - C_2 e^{-x}$

(see the rest on page 2)

2. (3 pt) Consider the equation

$$9x''(t) + cx'(t) + 4x(t) = 0.$$

(a) For which positive values of c do we have underdamping, critical damping, or overdamping? Explain.

(b) Take your favorite value of c for which we have underdamping and find two linearly independent solutions to the differential equation.

(a) The auxiliary equation is $9r^2 + cr + 4 = 0$.
Its roots are $r = \dfrac{-c \pm \sqrt{c^2 - 144}}{18}$. $144 = 12^2$

$c > 12$: $c^2 - 144 > 0$, 2 real roots → overdamping

$c = 12$: $c^2 = 144$, 1 real root → critical damping

$0 \le c < 12$: $c^2 - 144 < 0$, no real roots → underdamping

(b) Pick $c = 1$: $r = \dfrac{-1 \pm i\sqrt{143}}{18}$. =>

=> $x_1(t) = e^{-t/18}\cos\left(t\frac{\sqrt{143}}{18}\right)$,

$x_2(t) = e^{-t/18}\sin\left(t\frac{\sqrt{143}}{18}\right)$

(problem 1, continued)

Substitute initial values:

$1 = y(0) = c_1 + c_2$

$0 = y'(0) = -\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + c_1 - c_2 \Rightarrow c_1 - c_2 = \frac{1}{2}$.

$c_1 = \frac{1}{2}(1 + \frac{1}{2}) = 3/4$, $c_2 = \frac{1}{2}(1 - \frac{1}{2}) = 1/4$

Answer: $y = -\frac{1}{2}xe^{-x} - \frac{1}{2}x\cos x + \frac{1}{2}\sin x + \frac{3}{4}e^{x} + \frac{1}{4}e^{-x}$