18.156, SPRING 2017, PROBLEM SET 7, SOLUTIONS

1 (a) ⇒ (b) Fix χ and W as in (a). Take $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset W$ and $\psi(\xi_0) \neq 0$. Define

$$B := \psi(hD_x)\chi \in \Psi_h^0(\mathbb{R}^n).$$

We have for all ξ,

$$\widehat{Bu}(\xi/h) = \psi(\xi)\widehat{\chi u}(\xi/h). \tag{1}$$

Since (a) holds and ψ is compactly supported inside W, we have for all N

$$\int_{\mathbb{R}^n} \langle\xi\rangle^N |\widehat{Bu}(\xi/h)|^2\, d\xi = \mathcal{O}(h^\infty)$$

which implies that for all N

$$\|Bu\|_{H_h^N} = \mathcal{O}(h^\infty).$$

Now, since $\sigma_h(B) = \chi(x)\psi(\xi)$ (by the product formula) and $\chi(x_0)\psi(\xi_0) \neq 0$, the operator B is elliptic at (x_0, ξ_0). Choose a small neighborhood V of (x_0, ξ_0) such that $V \subset \operatorname{ell}_h(B)$, and assume that $A \in \Psi_h^0(\mathbb{R}^n)$ is compactly supported and $\operatorname{WF}_h(A) \subset V$. Using the elliptic parametrix construction, we construct $Q \in \Psi_h^0(\mathbb{R}^n)$ such that

$$A = QB + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

Applying this to u and using that u is h-tempered, we get for all N

$$\|Au\|_{H_h^N} \leq C\|Bu\|_{H_h^N} + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty).$$

Therefore, (b) holds.

1 (b) ⇒ (a) Fix V from (b). Choose $\chi, \psi \in C_c^\infty(\mathbb{R}^n)$ and a neighborhood W of ξ_0 such that

$$\chi(x_0) \neq 0, \quad \psi \equiv 1 \text{ on } W, \quad \operatorname{supp}\chi \times \operatorname{supp}\psi \subset V.$$

Put $B := \psi(hD_x)\chi$ as before. Fix $\chi_1 \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(1-\chi_1) \cap \operatorname{supp}\chi = \emptyset$. By pseudolocality of $\psi(hD_x)$ and h-temperedness of u we have for all N

$$\|(1-\chi_1)Bu\|_{L^2} = \mathcal{O}(h^\infty).$$

On the other hand, since $\chi_1 B$ is compactly supported and $\operatorname{WF}_h(\chi_1 B) \subset V$, we have by (b)

$$\|\chi_1 Bu\|_{L^2} = \mathcal{O}(h^\infty).$$

Therefore $\|Bu\|_{L^2} = \mathcal{O}(h^\infty)$, which by (1) implies (a).

2 (a) We have $Pu = 0$ where

$$P = hD_x - ix, \quad \sigma_h(P) = p, \quad p(x,\xi) = \xi - ix.$$

The symbol p is elliptic except at $(0,0)$, therefore by the elliptic estimate $\mathrm{WF}_h(u) \subset \{(0,0)\}$. The only problem is that p is not bounded uniformly in x, so to obtain a legal solution we can take $P = hD_x - i\psi(x)$ where $\psi \in C^\infty(\mathbb{R};\mathbb{R})$ satisfies

$$\psi(x) = \begin{cases} x, & |x| \ll 1; \\ \operatorname{sgn} x, & |x| \gg 1, \end{cases}$$

and $\psi \neq 0$ on $\mathbb{R} \setminus 0$. With this definition we have

$$f := Pu = i(x - \psi(x))e^{-\frac{x^2}{2h}}$$

and since $x - \psi(x)$ is supported away from zero and the Gaussian is rapidly decaying in h away from 0 we have $f = \mathcal{O}(h^\infty)_{\mathscr{S}(\mathbb{R})}$ and thus $\mathrm{WF}_h(f) = \emptyset$.

2 (b) We have $\mathrm{WF}_h(u) \subset \{x \geq 0\}$ since $\operatorname{supp} u \subset \{x \geq 0\}$. Next, we have $P_1 u = P_2 u = 0$ where

$$P_1 = \psi(x) \cdot hD_x, \quad \sigma_h(P_1) = p_1, \quad p_1(x,y,\xi,\eta) = \psi(x)\xi;$$
$$P_2 = hD_y, \quad \sigma_h(P_2) = p_2, \quad p_2(x,y,\xi,\eta) = \eta.$$

Here the function ψ is chosen as in part (a). By the elliptic estimate we have

$$\mathrm{WF}_h(u) \subset \{x\xi = 0\} \cap \{\eta = 0\}$$

and this gives the desired statement.

3. Since the statement that we need to prove is local and is near the fiber infinity, without loss of generality we may restrict to the cone $\mathcal{C} := \{\xi_1 \geq \varepsilon|\xi'|\} \subset \overline{T}^*\mathbb{R}^n$. Then we can use the coordinates suggested in the hint:

$$x_1, \ x', \ \rho := \xi_1^{-1}, \ \eta := \xi'/\xi_1.$$

Note that

$$\partial_{\xi_1} = -\rho^2\partial_\rho - \rho\eta \cdot \partial_\eta, \quad \partial_{\xi'} = \rho\partial_\eta.$$

In these coordinates the Hamiltonian vector field

$$H_p = \partial_{\xi_1} p \cdot \partial_{x_1} + \partial_{\xi'} p \cdot \partial_{x'} - \partial_{x_1} p \cdot \partial_{\xi_1} - \partial_{x'} p \cdot \partial_{\xi'}$$

takes the form

$$H_p = -(\rho\partial_\rho p + \eta \cdot \partial_\eta p)\rho\partial_{x_1} + \rho\partial_\eta p \cdot \partial_{x'} + \partial_{x_1} p \cdot \rho^2\partial_\rho + (\eta \cdot \partial_{x_1} p - \partial_{x'} p) \cdot \rho\partial_\eta.$$

By Exercise 3 in the previous problem set, the function $\langle\xi\rangle^{-k}p$ extends smoothly to the boundary of $\overline{T}^*\mathbb{R}^n$. In the region of interest we may replace $\langle\xi\rangle$ by ρ^{-1}, since $\rho\langle\xi\rangle = \sqrt{1 + \rho^2 + |\eta|^2}$ extends to a smooth nonvanishing function on $\mathcal{C}$. Therefore, $q := \rho^k p$ is a smooth function of x_1, x', ρ, η up to the fiber infinity $\{\rho = 0\}$. Similarly we may study the vector field $\rho^{k-1}H_p$ instead of $\langle\xi\rangle^{1-k}H_p$. We have

$$\rho^{k-1}H_p = -(\rho\partial_\rho q - kq + \eta \cdot \partial_\eta q)\partial_{x_1} + \partial_\eta q \cdot \partial_{x'} + \partial_{x_1} q \cdot \rho\partial_\rho + (\eta \cdot \partial_{x_1} q - \partial_{x'} q)\partial_\eta.$$

This is clearly a smooth vector field up to $\{\rho = 0\}$. Moveover, since $\rho^{k-1} H_p \rho = \rho \partial_{x_1} q$ vanishes on $\{\rho = 0\}$, the vector field $\rho^{k-1} H_p$ is tangent to the fiber infinity $\{\rho = 0\}$.