18.156, SPRING 2017, PROBLEM SET 7

1. This exercise shows that the two definitions of semiclassical wavefront set given in lecture are equivalent (at least away from fiber infinity). Let $u = u(h) \in \mathcal{D}'(\mathbb{R}^n)$ be a family of distributions, and assume for simplicity that $\|u\|_{L^2(\mathbb{R}^n)}$ is bounded uniformly in h (same would work for h-tempered u). Let $(x_0, \xi_0) \in \mathbb{R}^{2n} = T^*\mathbb{R}^n$. Show that the following two statements are equivalent:

(a) there exists $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, and a neighborhood W of ξ_0 such that

$$\int_W |\widehat{\chi u}(\xi/h)|^2 \, d\xi = \mathcal{O}(h^\infty); \tag{1}$$

(I replaced the L^∞ norm by L^2 here to make life a bit easier. With some more work one can show that this does not make a difference.)

(b) there exists a neighborhood V of (x_0, ξ_0) in $T^*\mathbb{R}^n$ such that

$$Au = \mathcal{O}(h^\infty)_{C^\infty} \quad \text{for all compactly supported } A \in \Psi_h^0(\mathbb{R}^n) \text{ with } \mathrm{WF}_h(A) \subset V.$$

Hint: To show (a) $\Rightarrow$ (b), note that (1) gives a bound on $\|Bu\|_{L^2}$ where $B = \psi(hD_x)\chi$ and $\psi \in C_c^\infty(W)$. If $A \in \Psi_h^0$ satisfies $\mathrm{WF}_h(A) \subset \{\chi \neq 0\} \times \{\psi \neq 0\}$, then $\|Au\|_{H_h^s}$ is controlled in terms of $\|Bu\|_{H_h^s}$ by the elliptic estimate, whose proof applies despite B not being a differential operator.

To show (b) $\Rightarrow$ (a), we can use the same operator B where now $\operatorname{supp}\chi \times \operatorname{supp}\psi \subset V$. The operator B is not compactly supported, however pseudolocality and boundedness on Sobolev spaces imply that $\|(1-\chi_1)B\|_{L^2 \to H_h^N} \leq C_N h^N$ for all N as long as $\chi_1 \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}(1-\chi_1) \cap \operatorname{supp}\chi = \emptyset$.

2. Show that

(a) for the Gaussian $u(x;h) = e^{-\frac{x^2}{2h}}$, $x \in \mathbb{R}$, we have $\mathrm{WF}_h(u) \subset \{(0,0)\}$;

(b) for the higher dimensional Heaviside function $u(x,y) = [x > 0]$, $(x,y) \in \mathbb{R}^2$,

$$\mathrm{WF}_h(u) \ \subset \ \{(x,y,0,0) \mid x \geq 0, \ y \in \mathbb{R}\} \cup \{(0,y,\xi,0) \mid y \in \mathbb{R}, \ \xi \in \overline{\mathbb{R}}\}.$$

Hint: the shortest (but not necessarily the easiest) solution is to find some operators $P \in \Psi_h^1$ such that $Pu = 0$ and use the elliptic estimate.

3. Assume that $p \in S^k(T^*\mathbb{R}^n)$. Show that the vector field $\langle\xi\rangle^{1-k} H_p$ extends to a smooth vector field on $\overline{T}^*\mathbb{R}^n$ which is tangent to the fiber infinity $\partial\overline{T}^*\mathbb{R}^n$. You may use Exercise 3 from the previous problem set. (Hint: write $\xi = (\xi_1, \xi')$, $\xi' \in \mathbb{R}^{n-1}$. If $\xi_1 \geq \varepsilon|\xi'|$ for some $\varepsilon > 0$, we can use the coordinate system $x_1, x', \rho := \xi_1^{-1}, \eta := \xi'/\xi_1$ on $\overline{T}^*\mathbb{R}^n$, with the boundary defining function ρ.)