18.156, SPRING 2017, PROBLEM SET 6, SOLUTIONS

1 (a) Denote $|g(x)| := |\det(g_{jk}(x))|$. Also, let $(g^{jk}(x))$ be the inverse matrix of $(g_{jk}(x))$. Then the Laplace–Beltrami operator Δ_g has the form

$$\Delta_g u(x) = |g(x)|^{-1/2} \sum_{j,k} \partial_{x_j}\big(|g(x)|^{1/2} g^{jk}(x)\partial_{x_k} u(x)\big) = \sum_{j,k} g^{jk}(x)\partial_{x_j}\partial_{x_k} u(x) + \dots$$

where '...' denotes a first order differential operator applied to u. Then

$$P = -h^2\Delta_g = -h^2 \sum_{j,k} g^{jk}(x)\partial_{x_j}\partial_{x_k} + h\Psi_h^1(\mathbb{R}^n).$$

It follows that $P \in \Psi_h^2(\mathbb{R}^n)$ and the principal symbol $p := \sigma_h(P)$ is given by

$$p(x,\xi) = \sum_{j,k} g^{jk}(x)\xi_j\xi_k. \tag{1}$$

Note that p is the square of the norm on covectors induced by the metric g.

1 (b) Assume that $(x(t),\xi(t)) = \varphi_t(x_0,\xi_0)$ for some fixed (x_0,ξ_0). Then $x(t),\xi(t)$ solve Hamilton's equations (with dots denoting derivatives in t)

$$\dot x_j(t) = (\partial_{\xi_j} p)\big(x(t),\xi(t)\big), \quad \dot\xi_j(t) = (-\partial_{x_j} p)\big(x(t),\xi(t)\big).$$

Using (1), we rewrite those as follows:

$$\dot x_j(t) = 2\sum_k g^{jk}(x(t))\xi_k, \tag{2}$$

$$\dot\xi_j(t) = -\sum_{k,\ell} (\partial_{x_j} g^{k\ell})\big(x(t)\big)\xi_k\xi_\ell. \tag{3}$$

Now, (2) gives immediately the equation for $2\xi_j(t)$ required in the problem.

It remains to show that $t \mapsto x(t)$ is a geodesic. For that we need to prove that $x(t)$ solves the geodesic equation

$$\ddot x_j(t) + \sum_{k,\ell} \Gamma^j_{k\ell}(x(t))\dot x_k(t)\dot x_\ell(t) = 0, \tag{4}$$

where $\Gamma^j_{k\ell}$ are the Christoffel symbols, given by

$$\Gamma^j_{k\ell} = \frac{1}{2}\sum_r g^{jr}\big(\partial_{x_k} g_{\ell r} + \partial_{x_\ell} g_{kr} - \partial_{x_r} g_{k\ell}\big). \tag{5}$$

Using (2), we rewrite (4) as

$$2\sum_k g^{jk}\dot\xi_k + 4\sum_{k,r,\alpha}(\partial_{x_r} g^{jk})g^{r\alpha}\xi_k\xi_\alpha + 4\sum_{k,\ell,\alpha,\beta} g^{k\alpha}g^{\ell\beta}\Gamma^j_{k\ell}\xi_\alpha\xi_\beta = 0. \tag{6}$$

Using (3) we rewrite (6) as

$$\sum_{k,\alpha,\beta} g^{jk}(\partial_{x_k} g^{\alpha\beta})\xi_\alpha\xi_\beta = 2\sum_{r,\alpha,\beta} g^{r\alpha}(\partial_{x_r} g^{j\beta})\xi_\alpha\xi_\beta + 2\sum_{k,\ell,\alpha,\beta} g^{k\alpha}g^{\ell\beta}\Gamma^j_{k\ell}\xi_\alpha\xi_\beta. \tag{7}$$

Since (g^{jk}) is the inverse matrix to (g_{jk}), we rewrite (7) as

$$\begin{aligned} &-\sum_{k,\alpha,\beta,r,\ell} g^{jk}g^{\alpha r}g^{\beta\ell}(\partial_{x_k} g_{r\ell})\xi_\alpha\xi_\beta \\ &= -2\sum_{k,\alpha,\beta,r,\ell} g^{jk}g^{\alpha r}g^{\beta\ell}(\partial_{x_r} g_{k\ell})\xi_\alpha\xi_\beta + 2\sum_{k,\ell,\alpha,\beta} g^{k\alpha}g^{\beta\ell}\Gamma^j_{k\ell}\xi_\alpha\xi_\beta. \end{aligned} \tag{8}$$

Finally, (8) follows from (5).

2. Denote $a := \sigma_h(A)$ and let $\mathrm{Char}(A)$ (the characteristic set) be the complement of $\mathrm{ell}_h(A)$ in the compactified cotangent bundle. Then:

(1) For $A = -h^2\Delta - 1$, we have

$$\begin{aligned} a(x,\xi) &= |\xi|^2 - 1, \\ \mathrm{Char}(A) &= \{|\xi| = 1\}, \\ e^{sH_a}(x,\xi) &= (x + 2s\xi, \xi). \end{aligned}$$

Note that $\mathrm{Char}(A)$ does not intersect the fiber infinity $\partial\overline{T}^*\mathbb{R}^n$.

(2) For $A = ih\partial_t - h^2\Delta_x$, denoting by $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ the momentum variables corresponding to t, x, we have

$$\begin{aligned} a(x,\xi) &= -\tau + |\xi|^2, \\ \mathrm{Char}(A) \cap T^*\mathbb{R}^n &= \{|\xi|^2 = \tau\}, \\ e^{sH_a}(t,x,\tau,\xi) &= (t - s, x + 2s\xi, \tau, \xi). \end{aligned}$$

To understand the intersection of $\mathrm{Char}(A)$ with the fiber infinity, introduce the following coordinate system valid for $(\tau,\xi) \neq 0$:

$$(\tau,\xi) = \rho^{-1}(\check\tau,\check\xi), \quad \rho \in [0,\infty), \quad (\check\tau,\check\xi) \in \mathbb{S}^n.$$

Note that ρ is a defining function of the fiber infinity, in particular $\partial\overline{T}^*\mathbb{R}^n = \{\rho = 0\}$. In this coordinate system, we have

$$\mathrm{Char}(A) = \{|\xi|^{-2}a(x,\xi) = 0\} = \{\check\xi^2 = \rho\check\tau\}.$$

Putting $\rho = 0$, we see that

$$\mathrm{Char}(A) \cap \partial\overline{T}^*\mathbb{R}^n = \{\check\xi = 0,\ \check\tau = \pm 1\}.$$

(3) For $A = h^2\partial_t^2 - h^2\Delta_x$, we have

$$a(x,\xi) = -\tau^2 + |\xi|^2,$$
$$\mathrm{Char}(A) \cap T^*\mathbb{R}^n = \{|\xi| = |\tau|\},$$
$$e^{sH_a}(t,x,\tau,\xi) = (t - 2s\tau, x + 2s\xi, \tau, \xi).$$

Moreover, in the coordinates introduced above

$$\mathrm{Char}(A) \cap \partial\overline{T}^*\mathbb{R}^n = \{|\check{\xi}| = |\check{\tau}| = 1/\sqrt{2}\}.$$

3. Both admitting a smooth extension to fiber infinity and having an asymptotic expansion in powers of $|\xi|$ are asymptotic questions as $|\xi| \to \infty$ (i.e. these properties trivially hold if a is compactly supported in ξ). Therefore we will restrict ourselves to $|\xi| \geq 1$.

Consider the polar coordinates in ξ:

$$\rho := |\xi|^{-1} \in (0,1), \quad \theta := \frac{\xi}{|\xi|} \in \mathbb{S}^{n-1}.$$

Note that (x,ρ,θ) extend to smooth coordinates on $\overline{T}^*\mathbb{R}^n$, with ρ being a defining function of the fiber infinity. We have

$$\partial_{\xi_k} = -\rho\theta_k\,\partial_\rho + \rho\Big(\partial_{\theta_k} - \sum_j \theta_k\theta_j\,\partial_{\theta_j}\Big).$$

The class $S^0_{1,0}$ consists of functions which are bounded under arbitrarily many applications of the vector fields $\partial_{x_1},\dots,\partial_{x_n},\rho^{-1}\partial_{\xi_1},\dots,\rho^{-1}\partial_{\xi_n}$. These fields give a frame for smooth vector fields which extend to the boundary of $\overline{T}^*\mathbb{R}^n$. Thus a smooth function on $\overline{T}^*\mathbb{R}^n$ is a symbol in $S^0_{1,0}$.

We now show that $S^k(T^*\mathbb{R}^n) = \langle\xi\rangle^k C^\infty(\overline{T}^*\mathbb{R}^n)$. Multiplying both sides by $|\xi|^{-k}$ and using that $|\xi|^{-k}\langle\xi\rangle^k = (1+\rho^2)^{k/2}$ is a smooth nonvanishing function on $\overline{T}^*\mathbb{R}^n$ (away from $\xi = 0$), we reduce to the case $k = 0$.

Note that positively homogeneous functions of order $j \in \mathbb{N}_0$ have the form $\rho^j a(x,\theta)$ where a is smooth. If $a \in C^\infty(\overline{T}^*\mathbb{R}^n)$, then using the Taylor expansion of a at $\rho = 0$ we obtain an asymptotic expansion in positively homogeneous functions and see that $a \in S^0(T^*\mathbb{R}^n)$.

On the other hand, let $a \in S^0(T^*\mathbb{R}^n)$. To show that a is smooth on $\overline{T}^*\mathbb{R}^n$, we note that each term in the asymptotic expansion for a is smooth (since it has the form $\rho^j a(x,\theta)$). Thus it suffices to consider the case when $a \in S^{-N}(T^*\mathbb{R}^n)$ and show $a \in C^{N-1}(\overline{T}^*\mathbb{R}^n)$. For any j,α,β, the function $\rho^{-N}(\rho\partial_\rho)^j\partial_x^\alpha\partial_\theta^\beta a$ is bounded. Taking $j < N$, we see that any order $< N$ derivative of a in x,ρ,θ is $\mathcal{O}(\rho)$. It follows that $a \in C^{N-1}(\overline{T}^*\mathbb{R}^n)$ and all order $< N$ derivatives of a vanish on $\{\rho = 0\}$.

4 (a) By induction we see that for each multiindices α, β the derivative $\partial_x^\alpha \partial_\xi^\beta (1/a)$ is a linear combination with constant coefficients of terms of the form

$$a^{-m-1}(\partial_x^{\alpha_1}\partial_\xi^{\beta_1} a)\cdots(\partial_x^{\alpha_m}\partial_\xi^{\beta_m} a) \tag{9}$$

where $\alpha_1 + \cdots + \alpha_m = \alpha$, $\beta_1 + \cdots + \beta_m = \beta$. Indeed, $1/a$ has the form (9) with $m = 0$ and if we differentiate (9) once in either x_j or ξ_j, we obtain a linear combination of terms of the form (9) (with updated α or β).

Now it remains to estimate each of the terms (9) using the derivative bounds on $a \in S^k_{1,0,h}$ and the ellipticity bound $|a| \geq c\langle\xi\rangle^k$:

$$\big|a^{-m-1}(\partial_x^{\alpha_1}\partial_\xi^{\beta_1} a)\cdots(\partial_x^{\alpha_m}\partial_\xi^{\beta_m} a)\big| \leq C\langle\xi\rangle^{-(m+1)k}\cdot\langle\xi\rangle^{k-|\beta_1|}\cdots\langle\xi\rangle^{k-|\beta_m|} = C\langle\xi\rangle^{-k-|\beta|}.$$

4 (b) We use Exercise 3, where we proved that $a \in S^k$ if and only if $b := \langle\xi\rangle^{-1}a$ admits a smooth extension to $\overline{T}^*\mathbb{R}^n$. Ellipticity of a implies that b is nonvanishing, thus $1/b$ is smooth on $\overline{T}^*\mathbb{R}^n$. Therefore, if $a \in S^k$ is elliptic, then $1/a \in S^{-k}$.

Now, assume that $a \in S^k_h$, namely

$$a \sim \sum_{j=0}^\infty h^j a_j, \quad a_j \in S^{k-j}.$$

We also assume that a is elliptic and thus a_0 is elliptic. Then $1/a_0 \in S^{-k}$, so that $a/a_0 \in S^0_h$. Since $1/a = (1/a_0)\cdot(a/a_0)^{-1}$, by replacing a by a/a_0 we may assume that $a_0 \equiv 1$. We then write $a = 1 - hq$ where $q \in S^{-1}_h$. Then we have $1/a \in S^0_h$, more precisely

$$1/a \sim \sum_{j=0}^\infty h^j q^j.$$

Indeed, we have for all J

$$1/a - \sum_{j=0}^{J-1} h^j q^j = h^J q^J/a.$$

Since $h^J q^J \in h^J S^{-J}_{1,0,h}$ and by part (a) $1/a \in S^0_{1,0,h}$, we have $h^J q^J/a \in h^J S^{-J}_{1,0,h}$, giving the asymptotic expansion.

5. We compute the principal symbol $p = \sigma_h(P)$:

$$p(x,\xi) = i\xi + 1.$$

Since $|p(x,\xi)| = \langle\xi\rangle$ it follows that $\mathrm{ell}_h(P) = \overline{T}^*\mathbb{R}$. Then the elliptic estimate gives the inequality required by the problem:

$$\|\chi_0 u\|_{L^2} = \mathcal{O}(h^\infty)\|\chi u\|_{L^2} \quad \text{when } Pu = 0. \tag{10}$$

Now, the set of solutions to $Pu = 0$ is spanned by the function

$$u = e^{-x/h}.$$

This function satisfies (10) because χ is chosen depending on χ_0, in particular we will always have $\operatorname{supp}\chi_0 \subset \{\chi = 1\}$, which implies that $|\chi| \geq 1/2$ on some neighborhood of $\operatorname{supp}\chi_0$. Therefore there exists $C = C(\chi_0, \chi) > 0$ such that

$$\|\chi_0 u\|_{L^2} \leq Ce^{-1/(Ch)}\|\chi u\|_{L^2}$$

and this gives (10).