18.156, SPRING 2017, PROBLEM SET 2

Recall that

$$P_V = -\partial_x^2 + V(x), \quad V \in C_c^\infty(\mathbb{R};\mathbb{R}),$$

and $R_V(\lambda) : C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is the scattering resolvent, in particular

$$(P_V - \lambda^2)R_V(\lambda)f = f \quad \text{for all } f \in C_c^\infty(\mathbb{R}).$$

The poles of $R_V(\lambda)$ are called resonances. Denote by $H^2(\mathbb{R})$ the Sobolev space with the norm

$$\|u\|_{H^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |u|^2 + |u'|^2 + |u''|^2\, dx.$$

Recall also that $C_c^\infty(\mathbb{R})$ is dense in both $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$.

1. Assume that $\operatorname{Im}\lambda > 0$ and λ is not a resonance.

(a) Using the integral formula for $R_V(\lambda)$ obtained in Problemset 1, Exercise 4, as well as Schur's inequality (see pages 2–3 of notes for lecture 3), show that $R_V(\lambda)$ extends to a bounded operator $L^2(\mathbb{R}) \to H^2(\mathbb{R})$. (Hint: first establish the boundedness $L^2 \to L^2$. For the first derivative, take $f \in C_c^\infty(\mathbb{R})$, $u = R_V(\lambda)f$, and integrate by parts in $\langle f, u\rangle_{L^2}$. For the second derivative, use the equation satisfied by u.)

(b) Show that $P_V - \lambda^2 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is invertible and $R_V(\lambda)$ is its inverse.

2*. Assume that λ is a resonance. Let $e_1 \in C^\infty(\mathbb{R})$ be a nontrivial outgoing solution to $(P_V - \lambda^2)e_1 = 0$ and fix any solution e_2 to the same equation which is linearly independent from e_1. We assume for simplicity that $W(e_1, e_2) = 1$ (this can always be arranged by multiplying e_2 by a constant).

(a) Show that the equation $(P_V - \lambda^2)u = f \in C_c^\infty(\mathbb{R})$ has an outgoing solution u if and only if

$$\langle f, \overline{e_1}\rangle_{L^2} = \int_{\mathbb{R}} f(x)e_1(x)\, dx = 0. \tag{1}$$

Show that under the condition (1), one outgoing solution is given by $u = R_1 f$ where

$$R_1 f(x) = \int_{\mathbb{R}} R_1(x,y)f(y)\, dy, \quad R_1(x,y) = e_1(x)e_2(y)[x > y] + e_2(x)e_1(y)[x < y].$$

(b) Fix $g, h \in C_c^\infty(\mathbb{R})$ such that $\langle g, \overline{e_1}\rangle_{L^2} = \langle e_1, h\rangle_{L^2} = 1$. Show that for each $f \in C_c^\infty(\mathbb{R})$, $\beta \in \mathbb{C}$, the *Grushin problem*

$$\begin{aligned}(P_V - \lambda^2)u + \alpha g &= f,\\ \langle u, h\rangle_{L^2} &= \beta\end{aligned} \tag{2}$$

has a unique solution $(u, \alpha) \in C^\infty(\mathbb{R}) \oplus \mathbb{C}$ such that u is outgoing. Show that this solution is given by the formulas

$$u = R_2 f + \beta e_1,$$
$$\alpha = \langle f, \overline{e_1} \rangle_{L^2}$$

where R_2 is the operator defined by

$$R_2 f = R_1 f - \langle f, \overline{e_1} \rangle_{L^2} \cdot R_1 g + \langle f, \overline{e_1} \rangle_{L^2} \langle R_1 g, h \rangle_{L^2} \cdot e_1 - \langle R_1 f, h \rangle_{L^2} \cdot e_1.$$

(c) Assume that $\operatorname{Im} \lambda > 0$. Show that R_2 extends to a bounded operator $L^2(\mathbb{R}) \to H^2(\mathbb{R})$. (Hint: compute the integral kernel of R_2 and argue similarly to Exercise 1(a). It seems that this part is rather painful.)

(d) Assume that $\operatorname{Im} \lambda > 0$. Show that the Grushin operator

$$\begin{pmatrix} P_V - \lambda^2 & g \\ h^* & 0 \end{pmatrix} : H^2(\mathbb{R}) \oplus \mathbb{C} \to L^2(\mathbb{R}) \oplus \mathbb{C}$$

is invertible, where $h^*(f) = \langle f, h \rangle_{L^2}$. Using this, show that $P_V - \lambda^2 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Fredholm operator of index 0. (Remark: the Fredholm property can be established in other ways, for instance by showing that the criterion for solvability in Exercise 2(a) applies to all $f \in L^2$. Later we will prove the Fredholm property using more advanced tools with much less pain.)