What we did last time:

$P_V = -\Delta + V$, $V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$, n odd.

$R_V(\lambda) = (P_V - \lambda^2)^{-1} : L^2 \to H^2$, $\operatorname{Im}\lambda > 0$

continues meromorphically to

$R_V(\lambda) : L^2_{comp} \to H^2_{loc}$, $\lambda \in \mathbb{C}$

Resonances = poles of $R_V(\lambda)$!

Some basic properties:

① $\forall f \in L^2_{comp}$, $(P_V - \lambda^2) R_V(\lambda) f = f$.
This is true for $\operatorname{Im}\lambda > 0$ by definition & continues analytically to all λ

② Defn: $u \in H^2_{loc}$ is outgoing at $\lambda \in \mathbb{C}$, if
$u = R_0(\lambda) g$ for some $g \in L^2_{comp}$,
where $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$...
Note: in this case $(P_V - \lambda^2) u$ ~~is comp~~ $\in L^2_{comp}$
Note: $u \in H^2_{comp} \Rightarrow u$ is outgoing at all λ, in fact
$u = R_0(\lambda)(-\Delta - \lambda^2) u$.
(true for $\operatorname{Im}\lambda > 0$ trivially; general λ by analytic continuation)

③ ~~Fact~~: If λ is not a resonance, then
$\forall f \in L^2_{comp}$, $R_V(\lambda) f$ is outgoing
Why? Recall how we constructed $R_V(\lambda)$:

$$R_V(\lambda) f = R_0(\lambda) \underbrace{(I + V R_0(\lambda)\rho)^{-1}(1 - V R_0(\lambda)(1-\rho)) f}_{\text{this is our } g!}$$

④ If λ is not a resonance, then
$\forall u \in H^2_{loc}$ outgoing at λ, $u = R_V(\lambda)(P_V - \lambda^2) u$
In particular, u outgoing, $(P_V - \lambda^2) u = 0 \Rightarrow u = 0$. **Exercise**

(5) Now assume λ_0 is a resonance, ~~$\lambda_0 \neq 0$~~
Also assume $\lambda_0 \neq 0$
(at λ_0, need more work, see §3.3) ~~(for $\lambda_0 = 0$, need more work, see §3.3)~~,

Write the Laurent expansion using λ^2 as the variable:

$$R_V(\lambda) = \underbrace{A_0(\lambda)}_{\substack{\text{holomorphic} \\ \text{near } \lambda_0}} + \sum_{j=1}^{J} \frac{A_j}{(\lambda^2-\lambda_0^2)^j}, \qquad A_j : L^2_{comp} \to H^2_{loc} \text{ finite rank}$$

Multiply by $(P_V - \lambda^2)$ on the left:

$$I = (P_V - \lambda^2) A_0(\lambda) + \sum_{j=1}^{J} \frac{(P_V - \lambda_0^2) A_j}{(\lambda^2-\lambda_0^2)^j} - \sum_{j=1}^{J} \frac{A_j}{(\lambda^2-\lambda_0^2)^{j-1}},$$

Equating the singular parts, we get

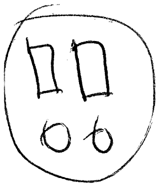
$$A_{j+1} = (P_V - \lambda_0^2) A_j, \quad (P_V - \lambda_0^2) A_J = 0.$$

Assuming J is chosen so that $A_J \neq 0$, denote

$\Pi_{\lambda_0} := A_J$. Then

$$R_V(\lambda) = A_0(\lambda) + \sum_{j=1}^{J} \frac{(P_V - \lambda_0^2)^{j-1} \Pi_{\lambda_0}}{(\lambda^2 - \lambda_0^2)^j}.$$

(5a) $(P_V - \lambda_0^2) A_J = 0$ and the range ~~of~~ of A_J consists of outgoing fns. [Exercise] Therefore, ~~$\lambda \neq 0$~~



~~is a~~ $\lambda_0 \neq 0$ a resonance

$\Updownarrow$

$\exists\, u \neq 0$, u outgoing, $(P_V - \lambda_0^2) u = 0$.

(5b) Semisimple case: $J = 1$.

Then Range $(\Pi_{\lambda_0}) = \{$outgoing solutions to $(P_V - \lambda_0^2) u = 0\}$ [Exercise]

In this case, define multiplicity of λ_0 = dim Range Π_{λ_0}.

In fact, in general ~~the~~ same f-la holds but multiplicities can get annoying.

A high frequency estimate

Want to bound $\|\chi R_V(\lambda)\chi\|_{L^2\to H^s}$, $s=0,1,2$

for $\chi\in C_c^\infty(\mathbb{R}^n)$,

asymptotically as $|\mathrm{Re}\,\lambda|\to\infty$, $\mathrm{Im}\,\lambda$ not too big.

Recall that if $\rho,\tilde\chi\in C_c^\infty$, $\rho V=V$, $\tilde\chi=1$ on $\mathrm{supp}\,\rho\cup\mathrm{supp}\,\chi$, then

$$(*)\ \chi R_V(\lambda)\chi=\chi R_0(\lambda)\tilde\chi\,(I+VR_0(\lambda)\rho)^{-1}(I-VR_0(\lambda)(1-\rho))\chi.$$

A bound on the cutoff free resolvent:

$$\|\rho R_0(\lambda)\rho\|_{L^2\to H^s}\le C_{\rho,L}(1+|\lambda|)^{-s}e^{L\,\widehat{(\mathrm{Im}\,\lambda)_-}}$$

$\widehat{\ }=\max(0,-\mathrm{Im}\,\lambda)$

where $L>0$ is such that $L>\mathrm{diam}\,\mathrm{supp}\,\rho$.

To prove this, we write

$$\rho R_0(\lambda)\rho=\int_0^L e^{i\lambda t}\rho\,U(t)\rho\,dt \text{ where}$$

$$U(t)=\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}},$$

and use estimates on $U(t)$.

(details: Theorem 3.1)

A bound on $R_V(\lambda)$: (Theorem 3.8) Given $A>0$,

There exist C, C_0, T (depending on ρ) such that for

$$\mathrm{Im}\,\lambda\ge -A-\delta\log(1+|\lambda|),\quad |\lambda|>C_0,$$

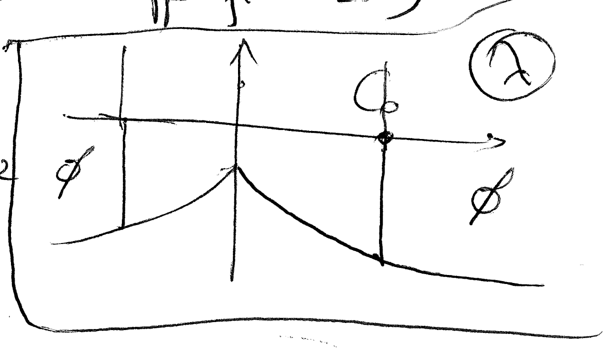
some δ, $0<\delta<\frac{1}{\mathrm{diam}\,\mathrm{supp}\,\rho V}$,

λ is not a resonance and

$$\|\rho R_V(\lambda)\rho\|_{L^2\to H^s}\le C|\lambda|^{s-1}e^{T(\mathrm{Im}\,\lambda)_-}.$$

Proof: we have (taking ρ, χ close to supp V, L such that $L\delta < 1$ but diam supp $\chi < L$)

$$\|V R_0(\lambda)\rho\|_{L^2 \to L^2}$$



$$\le C|\lambda|^{-1} e^{L(\operatorname{Im}\lambda)_-}.$$

However, for $|\lambda| \gg 1$, $(\operatorname{Im}\lambda)_- \le A + \delta \log|\lambda|$

So $e^{L(\operatorname{Im}\lambda)_-} \le C_A \cdot |\lambda|^{\delta L}$. So,

for $|\lambda| \gg 1$ we have $\|V R_0(\lambda)\rho\|_{L^2\to L^2} \le \frac{1}{2}$

and thus $\|(I + V R_0(\lambda)\rho)^{-1}\|_{L^2 \to L^2} \le 2$.

It remains to use $(*)$:

$$\|(I - V R_0(\lambda)(1-\rho))\chi\|_{L^2\to L^2} \le C e^{T(\operatorname{Im}\lambda)_-}$$

here T depends on χ

Similarly $\|\chi R_0(\lambda)\tilde{\chi}\|_{L^2 \to H^s} \le C|\lambda|^{s-1} e^{T(\operatorname{Im}\lambda)_-}$ ... □

Resonance expansion

Consider the wave eqn

$$(\star)\ \begin{cases} (\partial_t^2 + P_V) w(t,x) = 0, & t \ge 0,\ x \in \mathbb{R}^1 \\ w(0,x) = w_0(x) \in H^1_{\text{comp}}(\mathbb{R}^1) \\ \partial_t w(0,x) = w_1(x) \in L^2_{\text{comp}}(\mathbb{R}^1). \end{cases}$$

Thm. Assume $\operatorname{supp} w_j \subset B(0,K)$
$\chi \in C_c^\infty$, $\operatorname{supp} \chi \subset B(0,K)$,

and we're given $A > 0$.

Then there exist T, C such that (for some L_j)

$$w(t,x) = \sum_{\substack{\lambda_j \text{ resonance} \\ \operatorname{Im} \lambda_j \geq -A}} \sum_{\ell=0}^{L_j} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) + E_A(t)$$

where for $t \geq T$ we have

$$\|\chi E_A(t)\|_{H^2} \leq C e^{-tA} (\|w_0\|_{H^1} + \|w_1\|_{L^2}).$$

Proof. Will make a few simplifying assumptions to make it shorter...

(1) Assume $w_0 \equiv 0$, $w_1 \in C_c^\infty(\mathbb{R}^n)$.
Assume also that P_V has no resonances in $\operatorname{Im} \lambda > 0$,
that is the spectrum of P_V is inside $[0, \infty)$.

Using spectral theory, we write

$$w(t) = U(t) w_1, \quad U(t) = \frac{\sin(t\sqrt{P_V})}{\sqrt{P_V}}.$$

That is, $U(t) = F_t(P_V)$ where $F_t(\lambda) = \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}$ is entire.

<u>We have:</u> if $\hat{w}(\lambda) = \int_0^\infty e^{it\lambda} w(t)\, dt$, $\operatorname{Im} \lambda > 0$,

then
- $\|w(t)\|_{H^2} \leq Ct\|w_1\|_{H^1} \Rightarrow \hat{w}(\lambda) \in H^2$, $\operatorname{Im} \lambda > 0$
- Wave equation $\Rightarrow$ by IBP, $\widehat{\partial_t^2 w}(\lambda) = w_1 - \lambda^2 \hat{w}(\lambda)$

So, $(P_V - \lambda^2)\hat{w}(\lambda) = w_1$, $\hat{w}(\lambda) \in H^2$

Implies $\hat{w}(\lambda) = R_V(\lambda) w_1$.

By Fourier inversion formula applied to $t \mapsto e^{-t} w(t) [t>0]$, we get for $t>0$,

$$e^{-t} w(t) = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-it\lambda} \hat{w}(\lambda + i)\, d\lambda$$

i.e.
$$w(t) = \frac{1}{2\pi i} \int_{\operatorname{Im}\lambda = 1} e^{-it\lambda} \hat{w}(\lambda)\, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\operatorname{Im}\lambda = 1} e^{-it\lambda} R_V(\lambda) w_1 \, d\lambda .$$

~~(2) We now use the high frequency resolvent estimate.~~

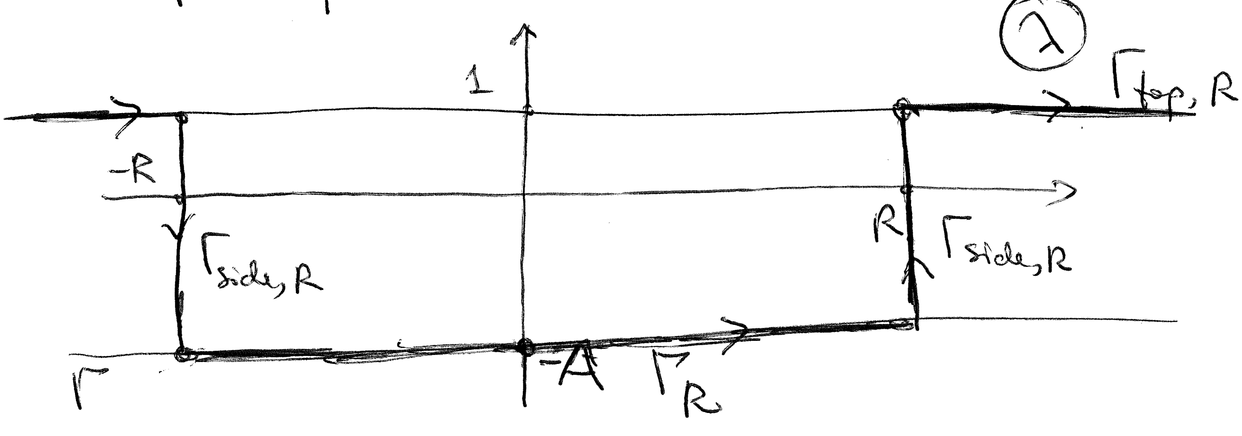
WLOG choose χ so that $w_1 = \chi w_1$, so that
$$\chi w(t) = \frac{1}{2\pi i} \int_{\operatorname{Im}\lambda = 1} e^{-it\lambda} \chi R_V(\lambda) \chi \, w_1 \, d\lambda .$$

(2) We now use the high frequency resolvent estimate:

$$\|\chi R_V(\lambda) \chi\|_{L^2 \to H^s} \leq C |\lambda|^{s-1} e^{T(\operatorname{Im}\lambda)_-}$$

when $\operatorname{Im}\lambda \geq -A - \delta \log(1 + |\lambda|)$, $|\lambda| > C_0$.

We will be lazier than the book and just deform to the contour $\Gamma = \{\operatorname{Im}\lambda = -A\}$ (assuming no resonances on Γ).



(3) For $R > C_0$, all resonances with $\operatorname{Im}\lambda \geq -A$ are contained in

$$[-R,R] + i[-A, 1].$$

By the residue Thm,

$$\chi w(t) = \sum_{\substack{\lambda_j \text{ res.} \\ \operatorname{Im}\lambda_j \geq -A}} \operatorname{Res}_{\lambda=\lambda_j} \left(e^{-it\lambda} \chi R_V(\lambda) \chi \right) w_1$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_{top,R} + \Gamma_{side,R} + \Gamma_R} e^{-it\lambda} \chi R_V(\lambda) \chi w_1 \, d\lambda.$$

For the Σ, write out the Laurent expansion at λ_j:

$$R_V(\lambda) = A(\lambda) + \sum_{\ell=1}^{J} \frac{A_{j\ell}}{(\lambda-\lambda_j)^\ell}.$$

On the other hand $e^{-it\lambda} = e^{-it\lambda_j} \sum_m \frac{(-it(\lambda-\lambda_j))^m}{m!}$

So the $(\lambda-\lambda_j)^{-1}$ terms in $e^{-it\lambda}\chi R_V(\lambda)\chi$ are given by a linear combination of $e^{-it\lambda_j} t^{\ell-1} \chi A_{j\ell} \chi$

Thus the

$$\Sigma = \sum_{\substack{\lambda_j \text{ res} \\ \operatorname{Im}\lambda_j \geq -A}} c_\ell \, e^{-it\lambda_j} t^{\ell-1} \chi A_{j\ell} \chi w_1.$$

(4) We finally analyse

$$\int_{\Gamma_{top,R}+\Gamma_{side,R}+\Gamma_R} e^{-it\lambda} \chi R_V(\lambda) \chi w_1 \, d\lambda.$$

The issue is with powers of λ: $\int (1+|\lambda|)^{-1}$ diverges ...

Note:

$$R_V(\lambda)(P_V - \lambda^2) \chi w_1 = \chi w_1.$$

Thus $\lambda^2 R_V(\lambda) \chi w_1 = R_V(\lambda) \underbrace{\chi P_V w_1}_{\text{still in } C_c^\infty} - w_1.$

So we're looking at

$$\int_{\Gamma_{top,R}+\Gamma_{side,R}+\Gamma_R} \frac{\chi R_V(\lambda) \chi P_V w_1 - \chi w_1}{\lambda^2} e^{-it\lambda} \, d\lambda.$$

The expression under the integral is, for $-A \leq \operatorname{Im}\lambda \leq 1$, $|\operatorname{Re}\lambda| > C_0$, bdd. in L^2 by $C_t |\lambda|^{-2}$.

Thus as $R \to \infty$, $\int_{\Gamma_{top,R}} \to 0$, $\int_{\Gamma_{side,R}} \to 0$,

$\int_{\Gamma_R} \to \int_{\Gamma}$. Remains to bound

$$E_A(t) := \int_{\Gamma} e^{-it\lambda} \underbrace{\chi R_V(\lambda) \chi w_1}_{\|\cdot\|_{H^1} \leq C|\lambda|^{-2} \|w_1\|_{H^2}} \, d\lambda$$

($\uparrow$ $|\cdot| = e^{-tA}$)

So, $\|E_A(t)\|_{H^1} \leq C \|w_1\|_{H^2}$.

For a better norm bound, need to go to the logarithmic contour. See Theorem 2.8 for details. $\square$