Semiclassical asymptotics (version 2)

$$P_V(h) = -h^2\partial_x^2 + V(x), \quad V \in C_c^\infty(\mathbb{R};\mathbb{R})$$

Interested in asymptotics of resonances as $h \to 0$.

Imagine that we have a sequence of resonances:

ω_j, $h_j \to 0$, $\omega_j = \omega(h_j)$, $\mathrm{Re}\,\omega_j \to \omega_\infty \in \mathbb{R}$,
$|\mathrm{Im}\,\omega_j| \le C_0 h$

& a corresponding sequence of resonant states

$$u = u(h_j), \quad \begin{cases} (P_V(h) - \omega_j^2)u = 0 \\ u \text{ outgoing, i.e. } u(x) \sim e^{\pm \frac{i\omega x}{h}}, \ \pm x \gg 1 \\ u(x) = O(1) \text{ locally in } x \text{ as } h \to 0 \end{cases}$$

1. Define $WF_h(u) \subset \mathbb{R}^2_{x,\xi}$, closed set, as follows:

$(x_0, \xi_0) \notin WF_h(u)$ if $\exists\, \chi \in C_c^\infty(\mathbb{R})$, $\chi(x_0) \neq 0$, such that $\widehat{\chi u}\left(\frac{\xi}{h}\right) = O(h^\infty)$ for ξ in some h-indepdt nbhd of ξ_0, uniformly in ξ.

Here we write $a(h) = O(h^\infty)$ if
$\forall N\ \exists C_N$ s.t. $|a(h)| \le C_N h^N$ for small enough h.

2. Define the principal symbol of $P_V(h)$,
$p(x,\xi) = \xi^2 + V(x)$.

Ellipticity ~~estimate~~: $WF_h(u) \subset \{(x,\xi) \mid p(x,\xi) = \omega_\infty^2\}$

3. Define the Hamiltonian flow $e^{tH_p} : \mathbb{R}^2_{x,\xi} \circlearrowleft$,
$H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi$, i.e.

$$e^{tH_p}(x_0,\xi_0) = (x(t), \xi(t)), \quad \begin{array}{l} x(0) = x_0, \ \xi(0) = \xi_0, \\ \dot{x}(t) = \partial_\xi p(x(t), \xi(t)), \\ \dot{\xi}(t) = -\partial_x p(x(t), \xi(t)). \end{array}$$

Propagation of singularities: $(x,\xi) \in WF_h(u) \Rightarrow e^{tH_p}(x,\xi) \in WF_h(u)$.

Note: $p(e^{tH_p}(x,\xi)) = p(x,\xi)$.

4. How to use the outgoing condition?

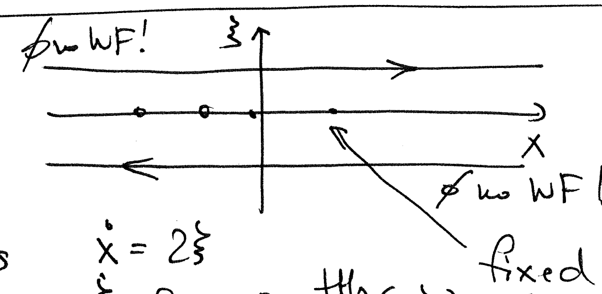
Say $u(x) = c_\pm e^{\pm \frac{i\omega x}{h}}$ for $\pm x \gg r_0$.

Then u lives at semiclassical frequency $\pm \operatorname{Re}\omega_\infty$ for $\pm x \gg r_0$.

So, $WF_h(u) \cap \{\pm x > r_0\} \subset \{\xi = \pm \omega_\infty\}$

(will be in pset in a ~~more~~ general form)

Examples: (A) $V \equiv 0$



Drawing the flow e^{tH_p} and the level sets of p:

$\dot x = 2\xi$

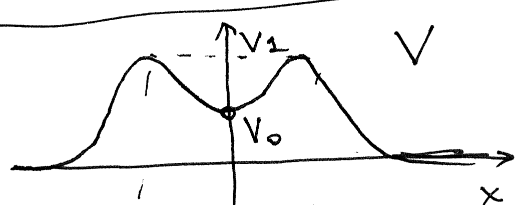
$\dot\xi = 0$ & $e^{tH_p}(x,\xi) = (x + 2t\xi, \xi)$

(2) + (3) + (4) $\Rightarrow$ for $\omega_\infty \neq 0$, no resonances!

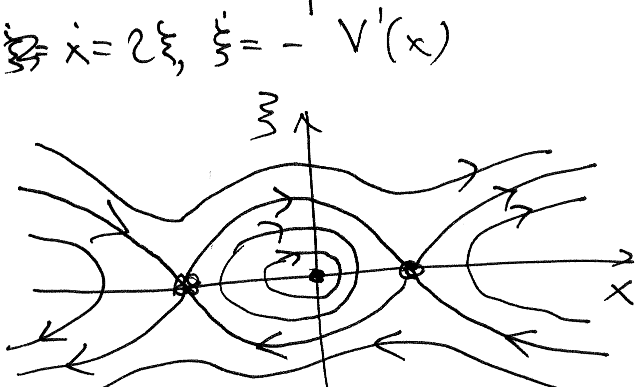
Also works for $h^2(-\partial_x^2 + \tilde V) = -h^2\partial_x^2 + h^2\tilde V$ ← lower order term

So this is the ~~nontrapping~~ ~~res~~ spectral gap we had before...

(B)



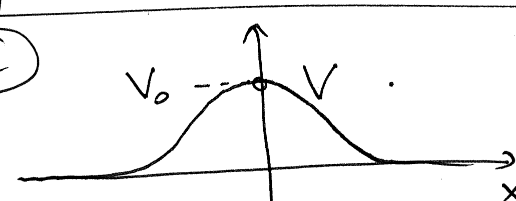
$\dot x = 2\xi$, $\dot\xi = -V'(x)$

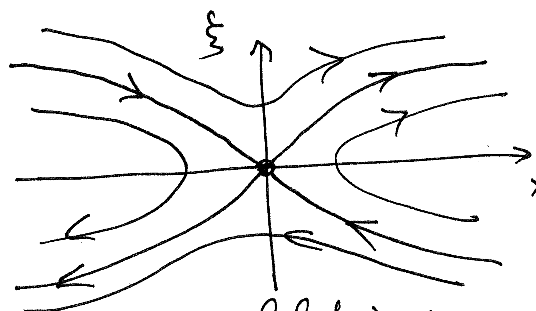


Can only have resonance sequences with $\sqrt{V_0} \le \omega_\infty \le \sqrt{V_1}$

Strong trapping → can get $\operatorname{Im}\omega \sim e^{-1/h}$ (least decaying)

(C)



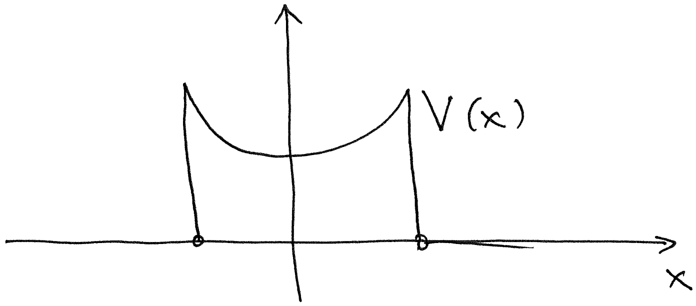


The only possibility is $\omega_\infty = \sqrt{V_0}$

Weak trapping → get $\operatorname{Im}\omega \sim h$ (least decaying)

MATLAB TIME

An (almost) explicit example:

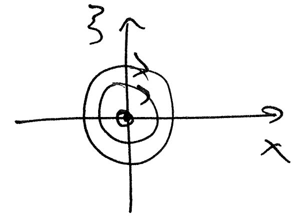


$$V(x) = \begin{cases} x^2+1, & -x<1<x \\ 0, & |x|>1 \end{cases}$$

$V \in L^\infty_{comp}$, not C_c^∞,

but resonances can still be defined. We'll use solutions to $(P_V - \lambda^2)u = 0$ which are C^1 at $x = \pm 1$.

Why x^2+1? Harmonic oscillator:

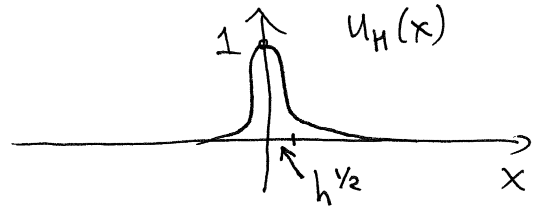


$p_H(x,\xi) = \xi^2 + x^2$

$P_H = -h^2 \partial_x^2 + x^2$

does not have a decaying potential... in fact has a complete basis of L^2 eigenfunctions, (Hermite Polynomials.. see Zworski, "Semiclassical Analysis" §6.1)

Ground state: $u_H(x) = e^{-\frac{x^2}{2h}}$.



We have $P_H u_H = h u_H$.

u_H gives almost a resonant state for $P_V(h)$ with $\tilde{\omega} = \sqrt{1+h}$, but does not satisfy the outgoing condition at $x = \pm 1$. However, we can find an actual resonance very close to $\tilde{\omega}$:

Thm For h small enough, $P_V(h)$ has a resonance of the form $\sqrt{z}$, $z = 1 + h - \frac{4i}{\sqrt{\pi}} h^{1/2} e^{-1/h}(1 + O(h))$.

Proof Not too involved but takes time, won't do it here. See §2.8 in the Book. $\square$

Potential scattering in higher (odd) dimensions

We consider the operator

$$P_V : \cancel{H^2} = -\Delta + V : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ and $V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$,

(for now actually $V \in L^\infty_{comp}$ would also work)

$n \geq 1$ odd.

Why odd? In even dimensions, meromorphic continuation is harder (log singularity at 0) and 3 is odd, so not so bad an assumption...

Want to define scattering resolvent

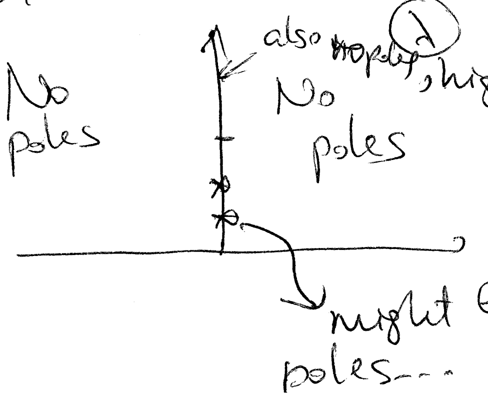
$$\boxed{R_V(\lambda).}$$

Upper half-plane.

Thm. $P_V - \lambda^2 : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $\operatorname{Im} \lambda > 0$ is Fredholm of index 0, and

$$R_V(\lambda) = (P_V - \lambda^2)^{-1} : L^2 \to H^2$$

exists when $\lambda^2 \notin [\inf V, \infty)$.



Spectrally speaking:
$P_V - \lambda^2$ self-adjoint, densely defined
Spectrum$(P_V) \subset [\inf V, \infty)$
Essential spectrum$(P_V) \subset [0, \infty)$.

Proof. ① $P_0 - \lambda^2 : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$
is invertible, for $\lambda > 0$, where $P_0 = -\Delta$.

Indeed, use the Fourier transform!

$$\widehat{(P_0-\lambda^2)u}(\xi) = (|\xi|^2-\lambda^2)\,\hat{u}(\xi)$$

and $u \mapsto \hat{u}$ is an isometry on L^2,

~~$H^2 =$~~ $\|u\|_{H^2} \sim \|(1+|\xi|^2)\hat{u}(\xi)\|_{L^2}$.

Can write $R_0(\lambda) = (P_0 - \lambda^2)^{-1} : L^2 \to H^2$

as follows: $R_0(\lambda)u = \mathcal{F}^{-1}\left(\frac{\hat{u}(\xi)}{|\xi|^2-\lambda^2}\right)$

inverse Fourier transform

where $\left|\frac{1}{|\xi|^2-\lambda^2}\right| \le \frac{C_\lambda}{1+|\xi|^2}$ since for $\lambda > 0$, so $\lambda^2 \notin [0,\infty)$.

② $V : H^2 \to L^2$ is compact by Rellich's Thm,
since V is compactly supported.

Thus for $\lambda > 0$, $P_V - \lambda^2 = \underbrace{P_0 - \lambda^2}_{\text{invertible}} + \underbrace{V}_{\text{compact}}$

is Fredholm of index 0.

③ P_V is self-adjoint, in particular
it is symmetric: $\forall u, w \in H^2(\mathbb{R}^n)$,

$$\langle P_V u, w\rangle_{L^2} = \langle u, P_V w\rangle_{L^2}$$

Why? true for $u, w \in C_c^\infty$ by integration by parts
and C_c^∞ is dense in H^2.

(4) If $\lambda \notin i\mathbb{R}$, $\operatorname{Im}\lambda > 0$, then $\operatorname{Im}(\lambda^2) \neq 0$. But if $P_V - \lambda^2 : H^2 \to L^2$ is not invertible, then by Fredholmness $\exists\, u \in H^2;\ u \neq 0,\ (P_V - \lambda^2)u = 0$.

Then
$$0 = \operatorname{Im} \langle (P_V - \lambda^2)u, u\rangle = -\operatorname{Im}(\lambda^2) \cdot \|u\|^2_{L^2},$$
a contradiction.

(5) If $\lambda \in i\mathbb{R}$, but $\lambda^2 \notin (\inf V, \infty)$; & $P_V - \lambda^2$ not invertible:
take u as in (4) & see that
$$0 = \langle (P_V - \lambda^2)u, u\rangle = \int_{\mathbb{R}^n} (-\Delta u + Vu - \lambda^2 u)\bar{u}\, dx$$
$$= \int |\nabla u|^2 + V|u|^2 - \lambda^2 |u|^2\, dx \geq \underbrace{(\inf V - \lambda^2)}_{>0} \|u\|^2_{L^2} > 0,$$
a contradiction. $\square$

Goal of the next few lectures:

<u>Thm</u>. $\forall \chi \in C_c^\infty(\mathbb{R}^n)$,
$$\chi R_V(\lambda) \chi : L^2 \to H^2 \text{ continues meromorphically}$$
from $\{\operatorname{Im}\lambda > 0\}$ to $\{\lambda \in \mathbb{C}\}$.

We usually say
$$R_V(\lambda) : \underset{\substack{\Uparrow \\ \text{compactly supported} \\ L^2 \text{ fns.}}}{L^2_{comp}} \to \underset{\substack{\Uparrow \\ \text{fns locally in } H^2:\ \forall \chi \in C_c^\infty, \\ \chi u \in H^2.}}{H^2_{loc}}$$