Scattering on hyperbolic surfaces

Def. A <u>hyperbolic surface</u> is a complete 2D Riemannian manifold (M, g) of (Gauss) curvature -1.

In the remaining 2 lectures we will talk about scattering on noncompact hyperbolic surfaces.

For more details, see the book of Borthwick (link on course website)

Classification of infinite ends

Let's first look for curvature -1 infinite ends in the form of a warped ~~stretched~~ product:

$$M = [a, \infty)_r \times \mathbb{S}^1_\theta \quad \text{where} \quad \mathbb{S}^1 = \mathbb{R}/\ell\mathbb{Z}, \ \ell > 0.$$

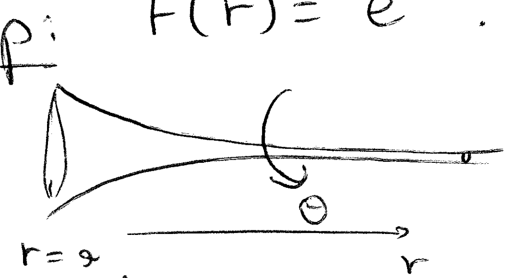
$$g = dr^2 + F(r)^2 d\theta^2.$$

One can compute the Gaussian curvature $= -\dfrac{F''}{F}$.

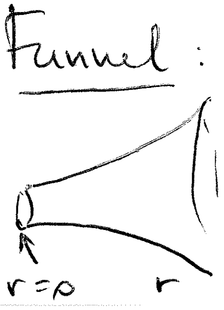
So, curvature $-1 \iff F'' = F$

Two important cases

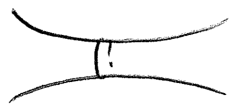
1. **Cusp:** $F(r) = e^{-r}$. Can assume $\ell = 1$ by ~~rescaling~~ shifting r...



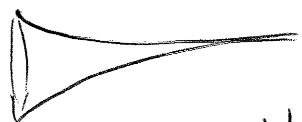
2. **Funnel:** $F(r) = \cosh r$, in our applications can always take $a = 0$ so that $\{r = 0\}$ is a closed geodesic ("neck")



<u>Note</u>: ~~$\mathbb{R}$, dr~~ $(\mathbb{R}\times S^1, dr^2+\cosh^2 r\, d\theta^2)$ is called the hyperbolic cylinder



and $(\mathbb{R}\times S^1, dr^2+e^{-2r}d\theta^2)$ is called the parabolic cylinder ("trumpet of death")



The hyperbolic space ~~itself~~ $\mathbb{H}^2$ can be written as ~~dr~~ $[0,\infty)_r \times S^1$, $g=dr^2+\sinh^2 r\, d\theta^2$, $\ell = 2\pi$ in "polar coordinates"

<u>Thm [Borthwick, Thm 2.23]</u> Assume that (M,g) is a connected hyperbolic surface which is "geometrically finite" and $M \not\simeq \mathbb{H}^2$, $M \not\simeq$ parabolic cylinder.

Then we can ~~write~~ decompose M into a compact part & finitely many cusp ends and/or funnel ends:

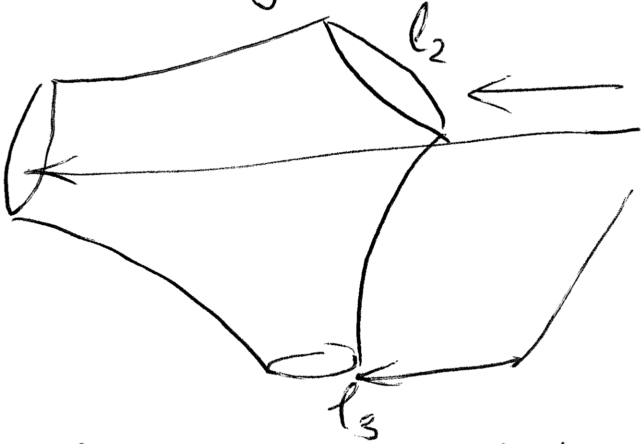
It will be convenient to henceforth assume that M is <u>oriented</u>.

<u>Note</u>: • M has no funnels – call it "finite area" (indeed, the area of M is finite)

• M has no cusps – call it "convex co-compact"

A complete description of the space of all possible ~~Riem~~ hyperbolic surfaces up to isomorphism (<u>moduli space</u>) is available via <u>Teichmüller theory</u> [Borthwick, §2.7]

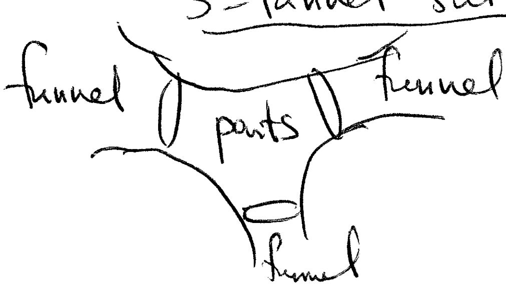
The building block is a <u>pair of pants</u>:



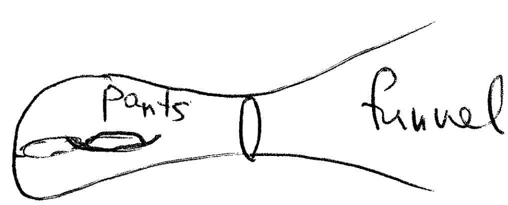
geodesic necks of lengths $\ell_j \geq 0$.
$\ell_j = 0 \to$ the neck is actually a cusp.

Any (oriented connected geometrically finite) hyperbolic surface can be obtained by gluing pairs of pants and possibly funnel ends along the necks, e.g. (convex co-compact examples)

<u>3-funnel surface</u>



<u>funneled torus</u>



Spectral properties

Thm [Borthwick, Chapter 7]

The Laplace - Beltrami operator

$$-\Delta_g : H^2(M) \to L^2(M), \quad \text{where}$$

$\cancel{H^2(M)}$ $L^2(M) = L^2(M, dVol_g)$

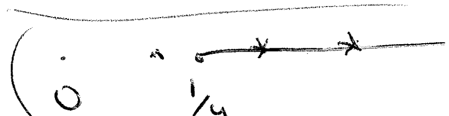
and $H^2(M)$ is defined using the metric g as well

$$\left(\|u\|^2_{H^2(M)} = \int_M |u|^2 + |\nabla_g u|^2 + |\nabla_g^2 u|^2 \, dVol_g \right)$$

is a nonnegative self-adjoint operator.

Its spectrum consists of:

- continuous spectrum $\cancel{[0, 1/4}$ $[1/4, \infty)$
- finitely many eigenvalues in $[0, \frac{1}{4})$
- (potentially) embedded eigenvalues in $[1/4, \infty)$



If M has infinite area (i.e. it has ≥ 1 funnel) then there are <u>no embedded eigenvalues</u>.

Meromorphic continuation

Thm [Borthwick, Thm 6.8, 6.11] The L^2 resolvent

$$R(\lambda) := (-\Delta_g - \tfrac{1}{4} - \lambda^2) : L^2 \to H^2, \quad \operatorname{Im} \lambda > 0$$

admits a meromorphic continuation with poles of finite rank

$$R(\lambda) : L^2_{comp} \to H^2_{loc}, \quad \lambda \in \mathbb{C}.$$

The poles of R are called <u>resonances</u>.

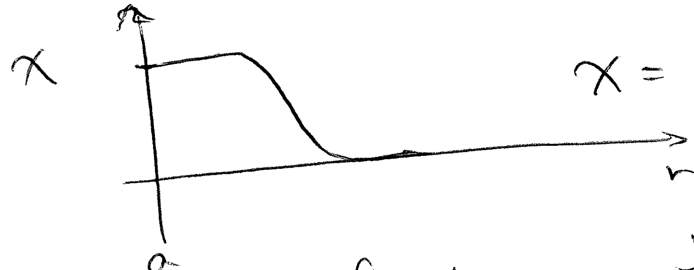
We'll explain the above theorems ~~in~~ the <u>finite area</u> case. $\boxed{\text{READ [Dy Zw, §4.1, 4.2 Example 3]}}$

Actually, let's assume that M has only 1 cusp.
We assume we already know that $\Delta_g : H^2 \to L^2$ is self-adjoint.

~~closure~~ C_c^∞ is dense in H^1...

In the cusp $[a,\infty)_r \times \mathbb{S}^1_\theta$, take

$$\chi \in C^\infty([a,\infty)), \quad \chi = 1 \text{ on } [a, a+1], \quad \chi = 0 \text{ on } [a+2, \infty)$$



$\chi = \chi(r)$, $\chi = 1$ outside of the cusp...

Decompose functions in the cusp into Fourier series, assuming for simplicity $\ell = 2\pi$,

$$u(r,\theta) = \sum_{k \in \mathbb{Z}} u_k(r) e^{ik\theta}.$$

Projection onto the 0 mode: $\Pi_0 : L^2(M) \to L^2(M)$,

$$\Pi_0 u = (1-\chi) u_0(r).$$

<u>Laplacian in the cusp:</u> recall that $g = dr^2 + e^{-2r} d\theta^2$, $d\mathrm{Vol}_g = e^{-r} dr\, d\theta$

So $-\Delta_g = -e^r(\partial_r e^{-r} \partial_r) - e^{2r}\partial_\theta^2$

$$= -\partial_r^2 + \partial_r - e^{2r}\partial_\theta^2.$$ So if ~~$Pu =$~~

$(-\Delta_g - \lambda^2 - \frac{1}{4}) u = f$ then

$$\left(-\partial_r^2 + \partial_r - \lambda^2 - \frac{1}{4} + e^{2r}k^2\right) u_k = f_k.$$

0 mode: $k=0$

$$\left(-\partial_r^2+\partial_r-\lambda^2-\frac{1}{4}\right)u_0=f_0.$$

The equation $\left(-\partial_r^2+\partial_r-\lambda^2-\frac{1}{4}\right)\check{u}=0$ has

2 solutions: $\cancel{r^{\frac{1}{2}+i\lambda},\ r^{\frac{1}{2}-i\lambda}}$. $e^{(\frac{1}{2}+i\lambda)r}$, $e^{(\frac{1}{2}-i\lambda)r}$

Recall that $d\mathrm{Vol}_g = e^{-r}\,dr\,d\theta$.

Thus $e^{(\frac{1}{2}+i\lambda)r}\in L^2([a,\infty)\times S^1, d\mathrm{Vol}_g)$

$\Updownarrow$

$\operatorname{Im}\lambda>0$

This explains the $1/4$ shift of λ^2...

So we expect scattering behavior in the 0 mode.

Model resolvent in the 0 mode: for $f\in L^2_{comp}(M)$

$\widetilde{R}_0(\lambda)f = (1-\chi(r))u(r)$ where

$$\begin{cases}\left(-\partial_r^2+\partial_r-\lambda^2-\frac{1}{4}\right)u=f & \text{on } [a,\infty)\\ u(r)\sim e^{(\frac{1}{2}+i\lambda)r} & r\gg 1\\ u|_{r=a}=0 & \text{(just to fix a unique solution } u)\end{cases}$$

Basic ODE analysis gives a holomorphic family

$$\widetilde{R}_0(\lambda):\begin{cases}L^2(M)\to H^2(M), & \operatorname{Im}\lambda>0\\ L^2_{comp}(M)\to H^2_{loc}(M), & \lambda\in\mathbb{C}.\end{cases}$$

$\neq 0$ modes: $\xrightarrow{k\neq 0}$ the potential $e^{2r}k^2$
is (exponentially) growing as $r\to\infty$.
So we expect that there is no scattering here.

Model resolvent: $(I-\Pi_0)(-\Delta_g+1)^{-1}$

Prop. $(I-\Pi_0)(-\Delta_g+1)^{-1} : L^2(M) \circlearrowleft$
is a compact operator.

Proof Imagine $f_j \in L^2(M)$ is a sequence
and $\|f_j\|_{L^2} \le 1$.

Take $u_j := (-\Delta_g+1)^{-1} f_j$. Since $-\Delta_g \ge 0$,
we get $\|u_j\|_{L^2} \le \|f_j\|_{L^2} \le 1$

Integrating by parts ~~(since C_c^∞ is dense~~
(i.e. using self-adjointness)
we get $\langle u_j, f_j\rangle_{L^2} = \int_M |\nabla_g u_j|^2 + |u_j|^2 \, d\mathrm{Vol}_g$.

Thus u_j is bdd in H^1:
$\|\nabla_g u_j\|_{L^2} \le \|f_j\|_{L^2} \le 1$.

To establish that $(I-\Pi_0)u_j$ is precompact, it
remains to get decaying bound in the cusp
as $r\to\infty$. We have for $u_j = \sum_k u_{jk}(r)e^{ik\theta}$,

~~$\|\nabla_g u_j\|^2_{L^2} = \sum_k$~~ if $\mathcal{C}$ denotes the cusp: $\mathcal{C} \simeq [a,\infty)_r \times S^1_\theta$,

then $\|\nabla_g u_j\|^2_{L^2(\mathcal{C})} = \int_a^\infty \int_{S^1} \left(|\partial_r u_j|^2 + e^{2r}|\partial_\theta u_j|^2\right) e^{-r}\, dr\, d\theta$

So we see that

$$1 \geq \|\nabla_g u_j\|^2_{L^2(\mathcal{C})} = \sum_{k\in\mathbb{Z}} \int_a^\infty \left(e^{-r}|\partial_r u_{jk}|^2 + k^2 e^r |u_{jk}|^2\right) dr.$$

So in the end, $\|\nabla_g u_j\|_{L^2} \leq 1$ and thus

$\|\nabla_g(1-\Pi_0) u_j\|_{L^2} \leq 1$, also

$\|e^r(1-\Pi_0)u_j\|_{L^2} \leq 1$ (where we extend r somehow into $M\setminus\mathcal{C}$)

Thus $(1-\Pi_0)u_j$ is precompact in L^2

$\Downarrow$

$(1-\Pi_0)(-\Delta_g+1)^{-1}$ is ~~pre~~ compact $L^2 \to L^2$ as needed. $\square$

How to prove meromorphic continuation of $R(\lambda)$?

Write $Q = \cancel{\chi_1(-\Delta_g} \; \chi_1(1-\Pi_0)(-\Delta_g+1)^{-1}\chi_2$

$$+(1-\chi_3)\tilde{R}_0(\lambda)(1-\chi_4)$$

for a correct choice of $\chi_1, \chi_2, \chi_3, \chi_4 \in C_c^\infty(M)$

So that $(-\Delta_g - \lambda^2 - \frac{1}{4})Q = I + Z(\lambda)$

where $Z(\lambda) : L^2 \to L^2$ is compact

& use Analytic Fredholm Theory...

(as in May 4 lecture)

Scattering operator

$\boxed{\text{READ [DyZw, § 4.4]}}$

We're looking for solution $E(\lambda) \in C^\infty(M)$ of

$$(*)\quad \begin{cases} \left(-\Delta_g - \lambda^2 - \frac{1}{4}\right) E = 0 \\ \text{non zero modes of } E \text{ in the cusp are in } L^2 \\ \text{the 0 mode: } E_0(r) = e^{(\frac{1}{2}-i\lambda)r} + S(\lambda) e^{(\frac{1}{2}+i\lambda)r} \end{cases}$$

for some $S(\lambda) \in \mathbb{C}$. We call $S(\lambda)$ the <u>scattering coefficient</u> (for several cusps it becomes a matrix)

- If λ <u>not a resonance</u>, then $(*)$ has a unique solution (similarly to the Euclidean case, we can use $R(\lambda)$ to construct E, see the March 23 lecture)

  The function E is often called the Eisenstein function/series (will see this later...)

  In general E, S are meromorphic...
- We have $S(-\lambda) = S(\lambda)^{-1}$ (immediate from $(*)$)
- If $S(-\lambda) = 0$ then $E(x, -\lambda)$ is a purely outgoing solution at λ $\Rightarrow$ λ is <u>a resonance</u>.

  How about the converse?
- If λ is a resonance, then $\exists u$ outgoing, $(-\Delta_g - \frac{1}{4} - \lambda^2) u = 0$. 2 cases:

  (a) $S(-\lambda) = 0$ & u is a multiple of $E(x, -\lambda)$

  (b) $u_0(r) \equiv 0 \Rightarrow u \in L^2$, so $\frac{1}{4} + \lambda^2$ is an L^2 <u>eigenvalue</u>.

  $-\Delta_g \geq 0$ self-adjoint $\rightarrow$ can only happen for $\lambda \in \mathbb{R} \cup i[-\frac{1}{2}, \frac{1}{2}]$