Previously we studied scattering ~~on~~ for

$$P = -\Delta + V, \quad V \in C_c^\infty(\mathbb{R}^n).$$

It is time to handle a stronger perturbation...

Scattering on manifolds with Euclidean ends

Basic case: on $\mathbb{R}^n$

$P = -\Delta_g$ where g is a Riemannian metric on $\mathbb{R}^n$:

$$g = \sum_{j,k=1}^{n} g_{jk}(x)\, dx_j\, dx_k, \quad (g_{jk}) \text{ is a symmetric positive definite matrix}$$

& g is Euclidean ~~at~~ near ∞:

if $r(x) := |x|$, $\cancel{x} \in \mathbb{R}^n$, then $\exists\, r_0$:

$$g = \sum_{j=1}^{n} dx_j^2 \quad \text{for} \quad \cancel{x} \quad r \geq r_0.$$

Advanced case: (M, g) is a Riemannian manifold with Euclidean infinite ends, i.e. there exists a continuous function $r: M \to \mathbb{R}$ and a number $r_0 > 0$ such that:

1. $\{r \leq r_0\} \subset M$ is compact
2. $\{r \geq r_0\} = \bigsqcup_{\ell=1}^{L} E_\ell$ where each (E_ℓ, g) is isometric to $(\mathbb{R}^n \setminus B(0, r_0), dx^2)$

and $r = |x|$ under this isometry.

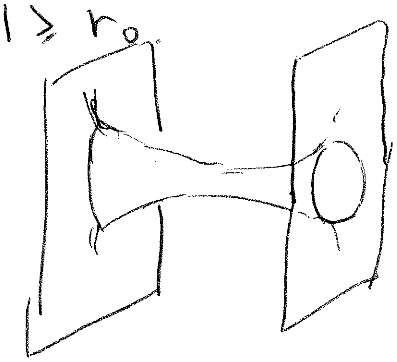
<u>Basic example</u>: manifold of revolution
(really, it's a stretched product...)

$M = (-\infty, \infty)_\rho \times \mathbb{S}^{n-1}_\theta$ and

$g = d\rho^2 + F(\rho)^2 d\theta^2$ where $d\theta^2$ is the round metric on $\mathbb{S}^{n-1}$ and $F \in C^\infty(\mathbb{R}; (0,\infty))$

satisfies $F(\rho) = |\rho|$ for $|\rho| \geq r_0$.

We can put $r = |\rho|$...



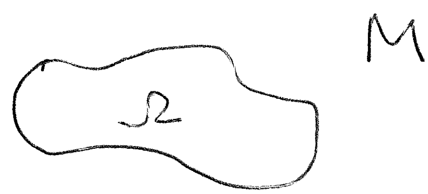
We can also make

M a manifold with C^∞ boundary,

and we require the boundary to be inside $\{r < r_0\}$
We impose Dirichlet boundary conditions...

<u>Example</u>: obstacle scattering

$M = \mathbb{R}^n \setminus \Omega$ where $\Omega \subset \mathbb{R}^n$ is a bdd domain with C^∞ bdary ~~e.g.~~:



We can also take

$P = -\Delta_g + V$ where $V \in C_c^\infty(M; \mathbb{R})$ ~~etc.~~

<u>Goals</u>:

- Obtain a meromorphic continuation of $(P - \lambda^2)^{-1}$ & define resonances
- Understand the high frequency distribution of resonances e.g. do we have a spectral gap?
Depends on the structure of trapped trajectories.

~~Meromo~~ Resolvent in the upper half-plane

We assume that $\partial M = \emptyset$.

Put $H^2(M) = \{u \in L^2(M) \mid \nabla u, \nabla^2 u \in L^2(M)\}$

where $L^2(M) = L^2(M; d\mathrm{vol}_g)$.

Then $P = -\Delta_g : H^2(M) \to L^2(M)$ bdd

Thm. Assume $\operatorname{Im} \lambda > 0$. Then

$$P - \lambda^2 : H^2(M) \to L^2(M)$$

is a Fredholm operator of index 0.

Proof For simplicity we assume $M \simeq \mathbb{R}^n$ but the proof works all the same in the general case.

We will construct $Q : L^2 \to H^2$ bdd

So that $I - (P-\lambda^2)Q$, $I - Q(P-\lambda^2)$ are compact.

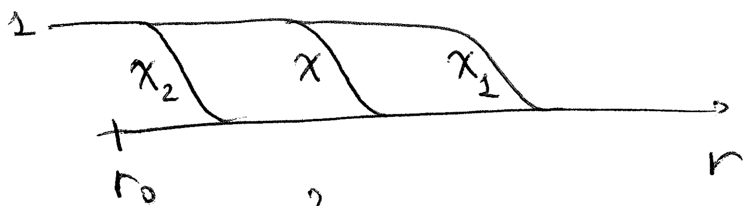
1 Fix cutoffs: choose $r_0 > 0$ s.t. $g = dx^2$ for $|x| \geq r_0$

Take $\chi, \chi_1, \chi_2 \in C_c^\infty(M)$ such that

$\chi_2 = 1$ near $\{r \leq r_0\}$,

$\chi = 1$ near $\operatorname{supp} \chi_2$, :

$\chi_1 = 1$ near $\operatorname{supp} \chi$



Imagine we want to solve for $f \in L^2$,

$Pu = f + \ldots$ where $\ldots$ has good regularity.

Enough to find u_1, u_2 : $\begin{cases} Pu_1 = \chi f + \cdots \\ Pu_2 = (1-\chi) f + \cdots \end{cases}$

$u := u_1 + u_2$

(2) To construct u_1, we use the elliptic parametrix

In fact, we can even go cheaper than that.

We use $Op := Op_1$, so we freeze $h := 1$,

$\Psi^k := \Psi^k_h$ with $h := 1$.

~~Take~~ We have $P = -\Delta_g = Op(p) + \Psi^1$ where

$$p(x,\xi) = \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k, \quad (g^{jk}) := (g_{jk})^{-1}.$$

Put $q(x,\xi) := \dfrac{\chi(x)}{p(x,\xi)+i} \in S^{-2}(T^*\mathbb{R}^n)$

Then $Op_h(q): L^2 \to H^2$. Also,

$P - \lambda^2 = Op(p+i) + \Psi^1$ (since $\lambda^2, i \in \Psi^0$)

So $(P-\lambda^2)\,Op_h(q) = \chi + \Psi^{-1}$,

$Op_h(q)\,(P-\lambda^2) = \chi + \Psi^{-1}$.

Put $u_1 := \chi_1\, Op_h(q) f$

(3) To construct u_2, we use the <u>free resolvent</u>

$R_0(\lambda) = (-\Delta-\lambda^2)^{-1}: L^2 \to H^2$ where $\Delta = \sum_j \partial_j^2$

(recall: $\widehat{R_0(\lambda) f}(\xi) = \dfrac{\hat{f}(\xi)}{|\xi|^2-\lambda^2}$. Here is where we use $\operatorname{Im}\lambda > 0$.)

Put $u_2 := (1-\chi_2)\, R_0(\lambda)\, (1-\chi) f$.

(4) Add things up: $u = u_1 + u_2 = Qf$ where

$$\boxed{Q = \chi_1\, Op_h(q) + (1-\chi_2)\, R_0(\lambda)(1-\chi).}$$

Now, we compute

$$(P-\lambda^2)Q = (P-\lambda^2)\chi_1 \mathrm{Op}_h(q) + (P-\lambda^2)(1-\chi_2)R_0(\lambda)(1-\chi$$

$$= \chi_1(P-\lambda^2)\mathrm{Op}_h(q) + [P,\chi_1]\mathrm{Op}_h(q)$$

$$+ (1-\chi_2)(\underbrace{P}_{-\Delta \text{ is fine too}}-\lambda^2)R_0(\lambda)(1-\chi) - [P,\chi_2]R_0(\lambda)(1-\chi)$$

$$= \overbrace{\chi_1\chi}^{\chi} + \chi_1\cdot\Psi^{-1} + [P,\chi_1]\mathrm{Op}_h(q)$$

$$+ \overbrace{(1-\chi_2)(1-\chi)}^{1-\chi} - [P,\chi_2]R_0(\lambda)(1-\chi)$$

$= I + Z_1(\lambda)$ where

$$Z_1(\lambda) = \chi_1\cdot\Psi^{-1} + [P,\chi_1]\mathrm{Op}_h(q) - [P,\chi_2]R_0(\lambda)(1-\chi)$$

maps $L^2 \to H^1$ with compact support

So by Rellich's Thm $Z_1 : L^2 \to L^2$ is a compact operator

Similarly

$$Q(P-\lambda^2) = \chi_1\mathrm{Op}_h(q)(P-\lambda^2) + (1-\chi_2)R_0(\lambda)(1-\chi)(P-\lambda^2)$$

$$= \overbrace{\chi_1\chi}^{\chi} + \chi_1\Psi^{-1} + \overbrace{(1-\chi_2)R_0(\lambda)(\underbrace{P}_{-\Delta}-\lambda^2)(1-\chi)}^{1-\chi}$$

$$+ (1-\chi_2)R_0(\lambda)[P,\chi]$$

$= I + Z_2(\lambda)$ where

$$Z_2(\lambda) = \chi_1\Psi^{-1} + (1-\chi_2)R_0(\lambda)[P,\chi]$$

maps $H^2 \to H^3$ with compact support for the first term

& H^2 locally $\to H^3$ for the second term.

So $Z_2(\lambda): H^2 \to H^2$ is compact. $\square$

What if M had boundary?
The situation is trickier. However, with some help from spectral theory we get the following stmt ~~for~~ the Dirichlet problem:

$$P-\lambda^2 : \mathcal{D}_P \to L^2(M) \text{ is Fredholm of idx } 0$$
for $\operatorname{Im}\lambda > 0$

where $\mathcal{D}_P = H^2(M) \cap H^1_0(M)$
$= \{u \in H^2(M) : u|_{\partial M} = 0\}$.

Note in both cases P is self-adjoint:

$$\langle (P)u, v\rangle_{L^2} = \langle u, (P)v\rangle_{L^2}$$

when $u, v \in \mathcal{D}_P$.
So $P-\lambda^2$ is invertible when $\operatorname{Im}\lambda > 0$, $\operatorname{Re}\lambda \neq 0$ by considering $(P-\lambda^2)u$ & ~~taking~~ $\operatorname{Im}\langle (P-\lambda^2)u, u\rangle$

Moreover, unless we have a potential,

$$\langle Pu, u\rangle \geq 0 \quad \text{for } u \in \mathcal{D}_P$$

since $\langle Pu, u\rangle = \int_M |\nabla u|^2 \, dVol_g$.

So then $P-\lambda^2$ is ~~also~~ invertible for all λ, $\operatorname{Im}\lambda > 0$.

Meromorphic continuation

From now on, we assume <u>n is odd</u>.

<u>Thm (see [DyZw, Theorem 4.4])</u> The operator

$$R(\lambda) := (P-\lambda^2)^{-1} : L^2 \to H^2 \text{ (or } \mathcal{D}_P), \ \operatorname{Im}\lambda > 0$$

admits a meromorphic continuation w/poles of finite rank

$$R(\lambda) : L^2_{comp} \to H^2_{loc}, \quad \lambda \in \mathbb{C}.$$

<u>Proof</u> 1 Pick χ, χ_1, χ_2 as in the previous Thm. Fix λ_0, $\operatorname{Im}\lambda_0 > 0$. Put (we change Q from the previous Thm)

$$Q := \chi_1 R(\lambda_0)\chi + (1-\chi_2) R_0(\lambda)(1-\chi).$$

For $\operatorname{Im}\lambda > 0$, we compute

$$\begin{aligned}
(P-\lambda^2)Q = {} & \chi_1 (P-\lambda^2) R(\lambda_0)\chi + [P,\chi_1] R(\lambda_0)\chi \\
& - [P,\chi_2] R_0(\lambda)(1-\chi) \\
& + (1-\chi_2)(P-\lambda^2) R_0(\lambda)(1-\chi) \\
= {} & \chi_1\chi + (\lambda_0^2-\lambda^2)\chi_1 R(\lambda_0)\chi + [P,\chi_1] R(\lambda_0)\chi \\
& + (1-\chi_2)(1-\chi) - [P,\chi_2] R_0(\lambda)(1-\chi) \\
= {} & I + Z(\lambda) \text{ where}
\end{aligned}$$

$$\begin{aligned}
Z(\lambda) = {} & (\lambda_0^2-\lambda^2)\chi_1 R(\lambda_0)\chi + [P,\chi_1] R(\lambda_0)\chi \\
& - [P,\chi_2] R_0(\lambda)(1-\chi) : L^2 \to L^2
\end{aligned}$$

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2) To ensure invertibility of $Z(\lambda)$ at one point, take $\lambda := \lambda_0$, $\underline{\lambda_0 := e^{\frac{i\pi}{4}}\alpha}$, $\alpha \in \mathbb{R}$, $\alpha \gg 1$:

$$Z(\lambda_0) = [P, \chi](P - i\alpha^2)^{-1}\chi - [P, \chi_2](-\Delta - i\alpha^2)^{-1}(1-\chi).$$

Since $\|(P - i\alpha^2)^{-1}\|_{L^2\to L^2} \le \frac{1}{\alpha^2}$,

$$\|(P-i\alpha^2)^{-1}\|_{L^2\to H^2} \le C\|(P-i\alpha^2)^{-1}\|_{L^2\to L^2} + \|P(P-i\alpha^2)^{-1}\|_{L^2\to L^2} \le C$$

we get $\|(P-i\alpha^2)^{-1}\|_{L^2\to H^1} \le \frac{C}{\alpha}$

Thus $\|Z(\lambda_0)\|_{L^2\to L^2} \le \frac{C}{\alpha}$ $\Rightarrow$ $\|Z(\lambda_0)\|_{L^2\to L^2} \le \frac{1}{2}$ for α large enough.

3) We have $R(\lambda) = (P-\lambda^2)^{-1} = Q(I + Z(\lambda))^{-1}$ for $\operatorname{Im}\lambda > 0$ whenever $Z(\lambda)$ is invertible.

One final cutoff: take $\chi_3 \in C_c^\infty(M)$ s.t. $\chi_3 = 1$ on supp χ_1 (and thus on supp χ_2)

Then $Z(\lambda) = \chi_3 Z(\lambda)$ $\Rightarrow$

$$\Rightarrow (I + Z(\lambda))^{-1} = (I + Z(\lambda)\chi_3)^{-1}(I - Z(\lambda)(1-\chi_3))$$

(multiply both sides by $I + Z(\lambda)$ on the right)

And $I + Z(\lambda)$ invertible $\Rightarrow$ $I + Z(\lambda)\chi_3$ invertible..

So we get for some λ in upper half-plane

$$(*)\quad R(\lambda) = Q(I + Z(\lambda)\chi_3)^{-1}(I - Z(\lambda)(1-\chi_3)).$$

4 Now let's consider arbitrary $\lambda \in \mathbb{C}$:
we get (using that $R_0(\lambda): L^2_{comp} \to H^2_{loc}$ since n odd)

$$Q : L^2_{comp} \to H^2_{loc}$$
$$I - Z(\lambda)(1-\chi_3) : L^2_{comp} \to L^2_{comp}$$
$$Z(\lambda) : L^2 \to H^1_{comp},$$ thus $Z(\lambda)$ is compact $L^2 \to L^2$
& holomorphic in λ.

By Analytic Fredholm Theory,
$(I + Z(\lambda))^{-1} : L^2 \to L^2$ is meromorphic with poles of finite rank

and since $Z(\lambda) = \chi_3 Z(\lambda)$ we see that
$$(I + Z(\lambda))^{-1} = I - Z(\lambda)(I + Z(\lambda))^{-1}$$
maps $L^2_{comp} \to L^2_{comp}$.

Using (*), we get that
$R(\lambda) : L^2_{comp} \to H^2_{loc}$ is meromorphic. $\square$

As before, we call poles of $R(\lambda)$ <u>resonances</u>.
To each resonance $\lambda \neq 0$ corresponds a resonant state $u \in H^2_{loc}(M)$ (in fact, $u \in C^\infty(M)$), $u|_{\partial M} = 0$ if needed,
and $(P - \lambda^2)u = 0$, u is outgoing i.e.
$\exists\, \varphi \in L^2(\mathbb{R}^n)$: $u = R_0(\lambda)\varphi$ for $|x| \gg 1$.
To see the outgoing condition, recall that $\forall f \in L^2_{comp}$
$$Qf = \underbrace{\chi_1 R(\lambda_0)\chi f}_{\text{c.s. in } H^1_0 \cap H^2} + \underbrace{(1-\chi_2)R_0(\lambda)(1-\chi)f}_{\text{outgoing, } 0 \text{ near } \partial M} \ldots$$

We have an ~~analo~~ extension of Rellich's Uniqueness Theorem to this case:

if $\lambda \in \mathbb{R} \setminus \{0\}$ is a resonance & u is a corresponding resonant state, then u is compactly supported.

In fact the same proof applies.

If M is connected (actually, enough to have no compact connected components)

then R.U.T. gives that there are no resonances on $\mathbb{R} \setminus \{0\}$.

This uses a more general unique continuation principle than the one we proved.

Can still define the scattering operator

$$S(\lambda) : L^2(\partial_\infty M) \to L^2(\partial_\infty M),$$

$$\partial_\infty M = \bigsqcup_{\ell=1}^{L} \mathbb{S}^{n-1}$$ where L is the # of infinite ends.