Today we prove the following

Propagation of Singularities

Let $U \subset \mathbb{R}^n$ be open, $P \in \Psi_h^1(\mathbb{R}^n)$ be a differential operator, $\sigma_h(P) = p \in S^1$ be real-valued, $\varphi_t := \exp(tH_p) : T^*\mathbb{R}^n \circlearrowleft$

Assume $A, B, B_1 \in \Psi_h^0(\mathbb{R}^n)$ are compactly supported in U

and $\forall (x,\xi) \in WF_h(A)$, $\exists\, T \geq 0$:

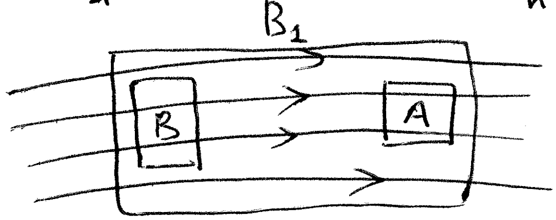
$\varphi_{-T}(x,\xi) \in \mathrm{ell}_h(B)$ and

$\varphi_{[-T,0]}(x,\xi) \in \mathrm{ell}_h(B_1)$.

Then $\exists\, \chi \in C_c^\infty(U)$ such that for all N

and all $u \in C^\infty(U)$,

$$\|Au\|_{L^2} \leq C\|Bu\|_{L^2} + Ch^{-1}\|B_1 Pu\|_{L^2} + C_N h^N \|\chi u\|_{H_h^{-N}}.$$



<u>Note</u>: this is weaker than PoS (II) we stated last time:

1. we assume $P \in \Psi_h^1$, i.e. $k=1$.
   also assume $s=0 \to$ estimates in L^2
   The general case is not harder though, just need to keep track of the orders s, k...
2. we assume $u \in C^\infty$ already $\boxed{\text{SEE EXERCISE E.28}}$
   (rather than $Bu \in L^2$, $B_1 f \in L^2 \Rightarrow Au \in L^2$ ...)

To get the regularity statement, need an extra regularization procedure that we do not do here...

Useful notation for the proof:

for an operator A, denote
$$\mathrm{Re}\,A := \frac{A+A^*}{2},\quad \mathrm{Im}\,A = \frac{A-A^*}{2i},$$ so that
$A = \mathrm{Re}\,A + i\,\mathrm{Im}\,A$ and $\mathrm{Re}\,A$, $\mathrm{Im}\,A$ are self-adjoint

Scheme of the proof: a positive commutator argument
(well, strictly speaking, negative commutator here...)

Step 1: fix $g \in C_c^\infty(\overline{T^*U};\mathbb{R})$ to be chosen later.
g is called escape function. Will have $\boxed{\mathrm{supp}\, g \subset \mathrm{ell}_h(B_1)}$.

Choose $G \in \Psi_h^0(\mathbb{R}^n)$ comp. supp. inside U,
$\boxed{\sigma_h(G) = g}$ (e.g. $G = \tilde{\chi}\,\mathrm{Op}_h(g)\,\tilde{\chi}$,
$\tilde{\chi} \in C_c^\infty(U)$, $\tilde{\chi}=1$ near ~~WF~~ $\mathrm{supp}\, g$)

WILL ARRANGE $\boxed{\mathrm{WF}_h(G) \subset \mathrm{ell}_h(B_1)}$!

Step 2: a formal computation.

$$\mathrm{Im}\langle Pu, G^*Gu\rangle_{L^2} = \mathrm{Im}\langle (\mathrm{Re}\,P)u, G^*Gu\rangle + \mathrm{Re}\langle (\mathrm{Im}\,P)u, G^*Gu\rangle \quad (\star)$$

Note: we can take $\langle\cdot,\cdot\rangle$ since
$u \in C^\infty \Rightarrow Pu \in C^\infty(U)$ and $G^*Gu \in C_c^\infty(U)$.
For general u, more work would be needed in this step!

LHS of $(\star)$: $|\mathrm{Im}\langle Pu, G^*Gu\rangle|$
$$\le \|GPu\|_{L^2}\cdot\|Gu\|_{L^2}$$
$$\le C\|B_1Pu\|_{L^2}\cdot\|Gu\|_{L^2} + O(h^\infty)\|\chi u\|_{L^2}^2$$ by elliptic estimate!

Step 3: analyze

$\mathrm{Re}\langle (\mathrm{Im}\, P) u, G^* G u\rangle$. We have

$$\sigma_h(P) = \overline{\sigma_h(P)} = \sigma_h(P^*) \Rightarrow \mathrm{Im}\, P \in h\Psi_h^{\circ}.$$

~~Thus $|\mathrm{Re}\langle (\mathrm{Im}\, P) u, G^* G u\rangle| \leq Ch \|u\|$~~

Write $\langle (\mathrm{Im}\, P) u, G^* G u\rangle = \langle G (\mathrm{Im}\, P) u, G u\rangle$

$$= \langle \underbrace{(\mathrm{Im}\, P)}_{\in h\Psi_h^{\circ}} G u, G u\rangle + \langle \underbrace{[G, \mathrm{Im}\, P]}_{\in h^2\Psi_h^{-1},\ \mathrm{WF} \subset \mathrm{ell}_h(B_1)} u, G u\rangle.$$

So, $|\mathrm{Re}\langle (\mathrm{Im}\, P) u, G^* G u\rangle|$

$$\leq C_1 h \|Gu\|_{L^2}^2 + Ch^2 \|B_1 u\|^2_{H_h^{-1/2}} + O(h^\infty)\|\chi u\|^2_{H_h^{-N}} \quad (1)$$

Step 4: analyze ~~Re~~ $\mathrm{Im}\langle (\mathrm{Re}\, P) u, G^* G u\rangle =$

$$= \frac{1}{2i}\left(\langle (\mathrm{Re}\, P) u, (G^* G) u\rangle - \langle (G^* G) u, (\mathrm{Re}\, P) u\rangle\right)$$

$$= \frac{1}{2i}\left(\langle G^* G (\mathrm{Re}\, P) u, u\rangle - \langle (\mathrm{Re}\, P) G^* G u, u\rangle\right)$$

$$= \frac{i}{2}\langle [\mathrm{Re}\, P, G^* G] u, u\rangle_{L^2}.$$ Got the commutator!

$$= h\langle Z u, u\rangle_{L^2} \text{ where } Z = \frac{i}{2h}[\underbrace{\mathrm{Re}\, P}_{\Psi_h^1}, \underbrace{G^* G}_{\Psi_h^{\circ}}] \in \Psi_h^{\circ}$$

compactly supp. in U.

By the commutator formula,

$$\sigma_h(Z) = \frac{1}{2}\{p, g^2\} = g \cdot H_p g \quad \text{(note: } H_p g = \{p, g\}\text{)}$$

We want $H_p g$ to have a specific sign so that this part is controlled.

This cannot be done everywhere since $g \in C_c^\infty$.

But we can do it everywhere *except* $\mathrm{ell}_h(B)$.

For that we come back to

Step 1 revisited: Construction of g

Lemma. Fix $C_1 > 0$ (will come from (1)).

Then $\exists\, g \in C_c^\infty(\overline{T}^*U;\mathbb{R})$ such that: supp $g \subset \mathrm{ell}_h(B_1)$ and

$\boxed{\text{READ [DyZw, LEMMA E.50]}}$

ⓐ $g \geq 0$ everywhere

ⓑ $g > 0$ on $WF_h(A)$

ⓒ $H_p g \leq -10 C_1 g$ in a nbhd of $\overline{T}^*U \setminus \mathrm{ell}_h(B)$.

Proof Imagine $WF_h(A) =$ one point (x_0, ξ_0).

Use the control condition: take $T > 0$ s.t.

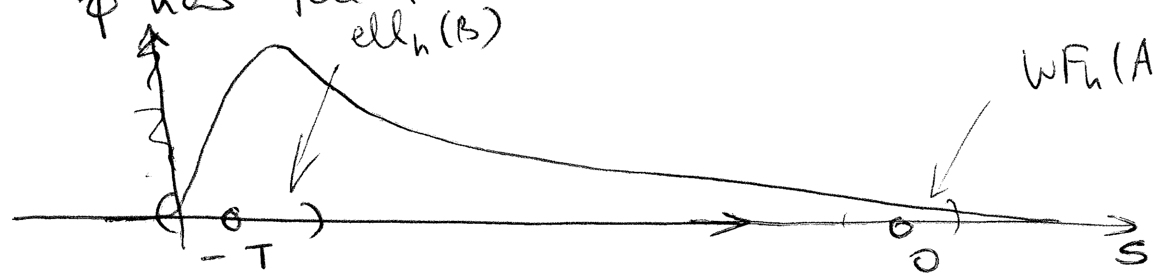
$\varphi_{-T}(x_0,\xi_0) \in \mathrm{ell}_h(B)$, $\varphi_{[-T,0]}(x_0,\xi_0) \in \mathrm{ell}_h(B_1)$.

$\boxed{(x_0,\xi_0) \mapsto (0,0)}$

Take coordinates (s,θ) near $\varphi_{[-T,0]}(x_0,\xi_0)$, such that $H_p = \partial_s$, accordingly $\varphi_t(s,\theta) = (s+t,\theta)$

Now put $g = \chi(s)\varphi(\theta)$ where $\varphi(0) = 0$, $\varphi \in C_c^\infty(\mathbb{R}^{n-1})$

and χ has the form:



Then if supp φ is small enough, g satisfies ⓐ – ⓒ

For general $WF_h(A)$, use partition of unity. $\square$

Back to Step 4. We have $\langle Zu, u\rangle$ and we know that

- $WF_h(Z) \subset \mathrm{ell}_h(B_1)$
- $\sigma_h(Z) = gH_p g \leq -10C_1 \sigma_h(G^*G)$ in a nbhd of $\overline{T}^*U \setminus \mathrm{ell}_h(B)$
- ~~$WF_h(A) \subset \mathrm{ell}_h(Z)$~~

To pass from $\sigma_h(Z) \leq \cdots$ to $\langle Zu, u\rangle \leq \cdots$ we use the following tool from semiclassical analysis:

Sharp Gårding Inequality $\boxed{\text{READ [Zw, Thm 4.32 + Thm 9.11]}}$

Assume $A \in \Psi_h^k(\mathbb{R}^n)$ and $\boxed{\operatorname{Re}\sigma_h(A) \geq 0.}$ Then $\exists\, C = C(A)$ s.t. $\forall\, u \in C_c^\infty(\mathbb{R}^n)$,

$$\boxed{\operatorname{Re}\langle Au, u\rangle_{L^2} \geq -Ch\|u\|_{H_h^{\frac{k-1}{2}}}}$$

Note: the meaning of A, B here is not the same as in the ambient proof.

Proof. Replacing A with $\frac{A+A^*}{2} = \operatorname{Re} A$, may assume $A^* = A$ and $a := \sigma_h(A) \geq 0$.

We will do the easy case $a = b^2$ where $b \in S^{\frac{k}{2}}$ real valued.

Put $B := \mathrm{Op}_h(b) \in \Psi_h^{\frac{k}{2}}$, then $\sigma_h(A) = \sigma_h(B^*B) \Rightarrow$

$\Rightarrow A = B^*B + hR$, $R \in h\Psi_h^{k-1}$. So then,

$\langle Au, u\rangle = \underbrace{\|Bu\|_{L^2}^2}_{\geq 0} + h\langle Ru, u\rangle$ and $|\langle Ru, u\rangle| \leq C\|u\|_{H_h^{\frac{k-1}{2}}}^2$

as $|\langle Ru, u\rangle| \leq C\|Ru\|_{H_h^{\frac{1-k}{2}}} \cdot \|u\|_{H_h^{\frac{k-1}{2}}}$

General case: see [Zw]. $\square$

Returning to Step 4:

$$\sigma_h(Z) + 10\, C_1 \sigma_h(G_1^* G) \le 0 \quad \text{on a nbhd of } T^+ \cup \text{ell}_h(B).$$

$$\sigma_h(B^*B) \ge c > 0 \quad \text{outside of this nbhd.}$$

So $\exists\, C_2$:

$$\sigma_h(Z) + 10\, C_1\, \sigma_h(G_1^* G) - C_2\, \sigma_h(B^*B) \le 0$$

everywhere!

~~Can actually replace B^*B with $(BB_1)^* BB_1$ since $WF_h(Z), WF_h(G) \subset \text{ell}_h(B_1)$.~~

~~So: $\sigma_h(Z + 10\, C_1 G^*G - C_2 (BB_1)^* BB_1) \le 0$ and $WF_h(\cdots) \subset \text{ell}_h$~~

So $\sigma_h(\underbrace{Z + 10 C_1 G^* G - C_2 B^* B}_{\tilde{Z}}) \le 0$ everywhere

Can shrink $WF_h(B)$ so that the control condition holds & $WF_h(\tilde{B}) \subset \text{ell}_h(B_1)$.

So then $WF_h(\tilde{Z}) \subset \text{ell}_h(B_1)$.

By sharp Gårding with an extra cutoff [EZ, Proposition E.35] $\exists\, C_3$ s.t.

$$\langle \tilde{Z} u, u \rangle \le \cancel{C_3 h \|u\|^2_{H_h^{1/2}}}\ C_3 h \|B_1 u\|^2_{H_h^{-1/2}} + O(h^\infty) \|\chi u\|^2_{H_h^{-N}}.$$

So

$$\langle Z u, u\rangle \le -10 C_1 \|Gu\|^2_{L^2} + C_2 \|Bu\|^2_{L^2} + C_3 h \|B_1 u\|^2_{H_h^{-1/2}} + O(h^\infty) \|\chi u\|^2_{H_h^{-N}}.$$

This gives

$$\mathrm{Im}\langle (\mathrm{Re}P)u, G^*Gu\rangle$$

$$\leq -10\, C_1 h \|Gu\|^2_{L^2} + Ch\|Bu\|^2_{L^2} + Ch^2\|B_1u\|^2_{H_h^{-1/2}} \quad (2)$$

$$+ O(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

~~&~~ Step 5

Combining (1), (2) see that (could take $C_1 \geq 1 \ldots$)

$$\mathrm{RHS}(\star) \leq -h\|Gu\|^2_{L^2} + Ch\|Bu\|^2_{L^2}$$

$$+ Ch^2\|B_1u\|^2_{H_h^{-1/2}} + O(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

So $\cancel{h}\|Gu\|^2_{L^2} \leq C\|Bu\|^2_{L^2} + Ch\|B_1u\|^2_{H_h^{-1/2}}$

$$+ Ch^{-1}\|B_1Pu\|_{L^2}\cdot\|Gu\|_{L^2} + O(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

Using $C\|B_1Pu\|_{L^2}\cdot\|Gu\|_{L^2} \leq \frac{1}{2}\|Gu\|^2_{L^2} + \tilde{C}\|B_1Pu\|^2_{L^2}$,

get $\|Gu\|_{L^2} \leq C\|Bu\|_{L^2} + Ch^{1/2}\|B_1u\|_{H_h^{-1/2}} + Ch^{-1}\|B_1Pu\|_{L^2}$

$$+ O(h^\infty)\|\chi u\|_{H_h^{-N}}.$$

Finally, $\mathrm{WF}_h(A) \subset \mathrm{ell}_h(B)$,

so by the elliptic estimate get

$$\|Au\|_{L^2} \leq C\|Bu\|_{L^2} + Ch^{-1}\|B_1Pu\|_{L^2} \quad (3)$$

$$+ Ch^{1/2}\|B_1u\|_{H_h^{-1/2}} + O(h^\infty)\|\chi u\|_{H_h^{-N}}.$$

Arguing similarly (just need to get the order of G right!) we get $\forall s$,

$$\|Au\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1Pu\|_{H_h^s} + Ch^{1/2}\|B_1u\|_{H_h^{s-\frac{1}{2}}} \quad (4)$$

$$+ O(h^\infty)\|\chi u\|_{H_h^{-N}}.$$

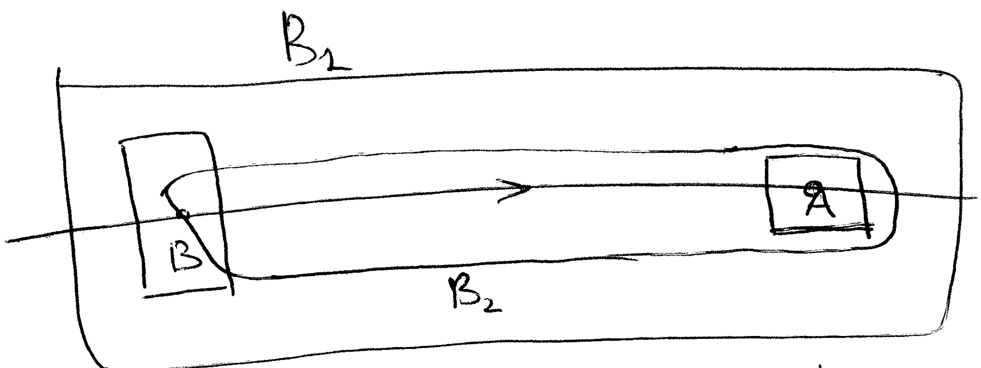
<u>Step 6</u>. It remains to remove the error $Ch^{1/2}\|B_1 u\|_{H_h^{s-\frac{1}{2}}}$. For that we <u>iterate</u>.

~~We prove by~~

Take A, B, B_1 satisfying the control condition & fix $B_2 \in \Psi_h^0(\mathbb{R}^n)$ [c.s. inside U]

So that A, B, B_2 satisfy c.c.

& B_2, B, B_1 satisfy c.c.



Then $\|Au\|_{H_h^s} \le C\|Bu\|_{H_h^s} + Ch^{-1}\|B_2 Pu\|_{H_h^s} + Ch^{1/2}\|B_2 u\|_{H_h^{s-\frac{1}{2}}} + O(h^\infty)$..

$$\le C\|Bu\|_{H_h^s} + Ch^{-1}\|B_2 Pu\|_{H_h^s} + Ch^{1/2}\left(C\|Bu\|_{H_h^{s-\frac12}} + Ch^{-1}\|B_1 Pu\|_{H_h^{s-\frac12}} + Ch^{1/2}\|B_1 u\|_{H_h^{s-1}}\right) + O(h^\infty)$$

can replace by B_1 by ellipticity

$$\le C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 Pu\|_{H_h^s} + Ch\|B_1 u\|_{H_h^{s-1}} + O(h^\infty)\|\chi u\|_{H_h^{-\infty}}$$

Iterating, we get $Ch^{j/2}\|B_1 u\|_{H_h^{s-j/2}}$ $\forall j$ which can be absorbed into $h^\infty\|\chi u\|_{H_h^{-N}}$.

End of proof of PoS. $\square$