Recall: if $u = u(h) \in \mathcal{D}'(U)$ is h-tempered, $U \Subset \mathbb{R}^n$, then $\mathrm{WF}_h(u) \subset \overline{T^*U}$ is defined as follows:

a point $(x_0, \xi_0) \in \overline{T^*U}$ does not lie in $\mathrm{WF}_h(u)$, iff

$\exists$ nbhd $V(x_0,\xi_0)$ such that $\forall A \in \Psi_h^0(\mathbb{R}^n)$ comp. supp. in U,

$$\mathrm{WF}_h(A) \subset V \Rightarrow \|Au\|_{H_h^N} \leq C_N h^N \quad \forall N.$$

Fundamental ~~Basic~~ question: Assume that $P \in \Psi_h^k(\mathbb{R}^n)$ is a differential operator, and $Pu = f$ where we know $\mathrm{WF}_h(f)$. What can we say about $\mathrm{WF}_h(u)$?

A better question is: what kind of microlocal estimates can we obtain on u?

BASIC EXAMPLE: fix $\eta \in \mathbb{R}$ & define

$$u(x;h) := e^{\frac{i\eta x}{h}}, \quad x \in \mathbb{R}.$$

We have $\mathrm{WF}_h(u) = \{\xi = \eta\} = \{(x,\xi) \in \overline{T^*\mathbb{R}} : \xi = \eta\}$. $(*)$

For our particular very special example we can use oscillatory testing: $a \in S^0 \Rightarrow \mathrm{Op}_h(a)u(x) = a(x,\eta)u(x)$.

This is $O(h^\infty)_{C^\infty} \Leftrightarrow a(x,\eta) = 0$ for all x...

This recovers $(*)$

A more conceptual way to see the $\subset$ containment in $(*)$ is to use the fact that

$$Pu = 0 \quad \text{where} \quad P = hD_x - \eta \in \Psi_h^1(\mathbb{R}).$$

Recall general elliptic estimate:

$$Pu = f \Rightarrow \mathrm{WF}_h(u) \subset \mathrm{WF}_h(f) \cup \mathrm{Char}_h(P)$$

where $\mathrm{Char}_h(P) = \overline{T^*U} \setminus \mathrm{Ell}_h(P) = \{\langle\xi\rangle^{-k}\sigma_h(P) = 0\}$.

For our basic example,
we have $p = \sigma_h(P)$, $p(x,\xi) = \xi - \lambda$, so
$\mathrm{Char}_h(P) = \{\xi = \lambda\}$...

We now obtain more information on how $WF_h(u)$ is distributed on $\mathrm{Char}_h(P)$.

Henceforth we make the following

ASSUMPTIONS:

$\boxed{u \in \mathcal{D}'(U) \text{ and } Pu = f \in \mathcal{D}'(U)}$

1. $U \Subset \mathbb{R}^n$ is open;
2. $P \in \Psi_h^k(\mathbb{R}^n)$ is a differential operator (our statements are local $\Rightarrow$ it's enough to ask that P be properly supported on U, and there is no need to define P outside of U)
3. $p := \sigma_h(P) \in S^k(T^*\mathbb{R}^n)$ is real-valued (can be relaxed to a sign condition, see the book...)

HAMILTONIAN FLOW:

- Let $H_p = \sum_{j=1}^n (\partial_{\xi_j} p \cdot \partial_{x_j} - \partial_{x_j} p \cdot \partial_{\xi_j})$ be the Hamiltonian vector field of p on $T^*\mathbb{R}^n$
- Let $\exp(tH_p) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be the Hamiltonian flow:
  $\exp(tH_p)(x_0,\xi_0) = (x(t), \xi(t))$ where $x(t), \xi(t)$ solve the ODE

$$\begin{cases} x(0) = x_0, \quad \xi(0) = \xi_0 \\ \dot{x}_j(t) = \partial_{\xi_j} p(x(t), \xi(t)) \\ \dot{\xi}_j(t) = -\partial_{x_j} p(x(t), \xi(t)) \end{cases}$$

Denote $\varphi_t = \exp(tH_p)$ [will change later]

PROPAGATION OF SINGULARITIES (I)

(easy to state, but often too weak to use)

Let u be h-tempered and $\boxed{f \equiv 0}$ (i.e. $Pu = 0$)

Then $WF_h(u)$ is invariant under the flow φ_t in U,

i.e. if $(x_0, \xi_0) \in WF_h(u)$, then

$\varphi_t(x_0, \xi_0) \in WF_h(u)$ as long as $\varphi_s(x_0, \xi_0) \in U$ for all s between 0 and t.

Remarks.

1) Recall that ellipticity gives
$WF_h(u) \subset \mathrm{Char}_h(P)$.
Conveniently, φ_t preserves $\mathrm{Char}_h(P)$
i.e. H_p is tangent to $\mathrm{Char}_h(P)$.

2) For our basic example:
$P = hD_x - \nu$, $p(x, \xi) = \xi - \nu$,
$H_p = \partial_x$, so $\varphi_t(x, \xi) = (x + t, \xi)$.
So, $Pu = 0 \Rightarrow WF_h(u) = \emptyset$ or $WF_h(u) = \{\xi = \nu\}$.

3) A small <u>problem</u> however:
how to extend φ_t to $\overline{T^*M}$, i.e. to fiber infinity?
For instance if $p = \xi^2$ instead of $p = \xi - \nu$, then
$\exp(tH_p)(x, \xi) = (x + 2t\xi, \xi)$. Does not extend well for $|\xi| = \infty$.

There is a simple solution: just rescale H_p: we <u>redefine</u>

$\boxed{\varphi_t := \exp(t\langle\xi\rangle^{1-k} H_p)}$ where $P \in \Psi_h^k$.

Then $\langle\xi\rangle^{1-k} H_p$ extends as a C^∞ vector field to $\overline{T^*\mathbb{R}^n}$ & it is tangent to the fiber infinity $\partial\overline{T^*M}$.

~~4 A classical example: $P = h^2(\partial_t^2 - \Delta_x)$ on $\mathbb{R}^4_{t,x}$.~~

~~$p(x,\xi,\tau, p(t,x,\tau,\xi) = -\tau^2 + |\xi|^2$~~

What happens for nonzero f?

PROPAGATION OF SINGULARITIES II

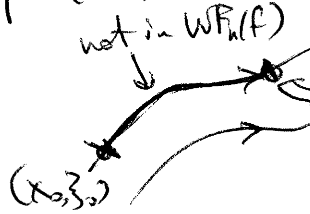
Let u be h-tempered & assume that $(x_0,\xi_0) \in \overline{T^*U}$ and $T \in \mathbb{R}$ satisfy:

(a) $(x_0,\xi_0) \notin WF_h(u)$

(b) for all t between 0 and T, we have $\varphi_t(x_0,\xi_0) \in \overline{T^*U}$ but $\varphi_t(x_0,\xi_0) \notin WF_h(f)$.

Then $\varphi_T(x_0,\xi_0) \notin WF_h(u)$.

Picture:



the flow φ_t

not in $WF_h(u)$ then

So, a singularity was either there in the past or it appeared along the way from the RHS

Note: P_oS II $\Rightarrow$ P_oS I trivially

A classical example: $P = h^2(\underbrace{\partial_t^2 - \Delta_x}_{\Box})$ on $\mathbb{R}^{n+1}_{t,x}$.

We have $p(t,x,\tau,\xi) = -\tau^2 + |\xi|^2$ and

$\mathrm{Char}_h(P) = \{ |\tau| = |\xi| \}$,

~~$\exp(sH_p)$~~ $\exp(sH_p)(t,x,\tau,\xi) = (t - 2s\tau, x + 2s\xi, \tau, \xi)$

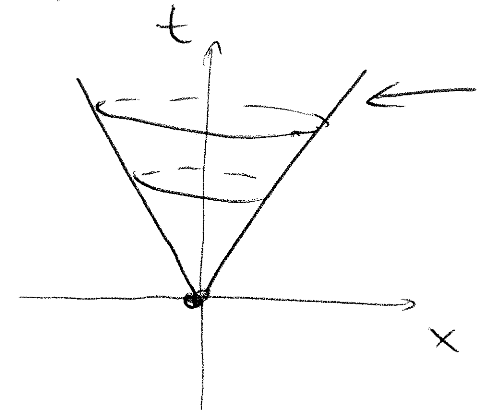
Classic solution is the forward fundamental solution:

$u \in \mathcal{D}'(\mathbb{R}^{n+1})$, $\mathrm{supp}\, u \subset \{t \geq 0\}$, ~~$\Box$ R~~ $\Box$

$\Box u = \delta(t)\delta(x)$. It has singularities (lack of being in C^∞)

on the cone ~~$\{(t,x,\tau,\xi) \mid t = |x|\}$~~,

$\{(t,x) : t = |x|\}$



← singularities (lack of smoothness of u) live here in <u>position space</u>.

What does <u>PoS (II)</u> give us?

$WF_h(\delta f) = WF_h(\delta(t)\delta(x)) = \{(t,x,\tau,\xi) \in T^*\mathbb{R}^n : t=0, x=0\}$

Then $WF_h(f) \cap \mathrm{Char}_h(P)$ induces singularities which propagate forward along the flow φ_t. We set (since $u=0$ for $t<0$, thus $WF_h(u) \subset \{t \geq 0\}$)

$WF_h(u) \subset \{(t,x,\tau,\xi)$: <u>either</u> $t=x=0$ <u>or</u> $(t=|x|>0$ & $\xi = -\tau \cdot \frac{x}{t})$ <u>or</u> $\xi=0, \tau=0\}$.

This is compatible with u having a jump discontinuity across the cone $\{t=|x|\}$. → corresponds to $u \in C^\infty$ inside the cone e.g. $n=2$...

<this is likely confusing because I could not afford to spend more time on this...>

But note that unlike the formula for the fundamental solution, propagation of singularities applies to variable coefficient equations e.g. $\Box \to \partial_t^2 - \Delta_g$ or even wave equation on a Lorentzian manifold (in general relativity).

So propagation of singularities is a mathematical version of <u>geometric optics</u>:

"light propagates along straight lines"

P.S II is nice & quite useful but it would be better if we had an <u>estimate</u>:

PROPAGATION OF SINGULARITIES III

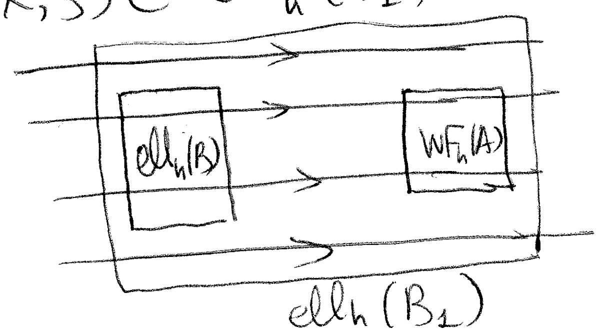
(a stronger statement, this is the one we (mostly) prove)

$\boxed{\text{READ [DyZw, Theorem E.49 \& Exercise E.28]}}$

Assume that $A, B, B_1 \in \Psi_h^0(\mathbb{R}^n)$ are compactly supported inside U and the following <u>control condition</u> holds:

for each $(x,\xi) \in WF_h(A)$ there exists $T \geq 0$ such that $\varphi_{-T}(x,\xi) \in \mathrm{ell}_h(B)$, and $\varphi_t(x,\xi) \in \mathrm{ell}_h(B_1)$ for all $t \in [-T, 0]$.

<u>Picture</u>:



Assume that (for some s) $u \in \mathcal{D}'(U)$ and $Bu \in H^s_{\cancel{loc}}$, $B_1 f \in H^{s-k+1}_{loc}$

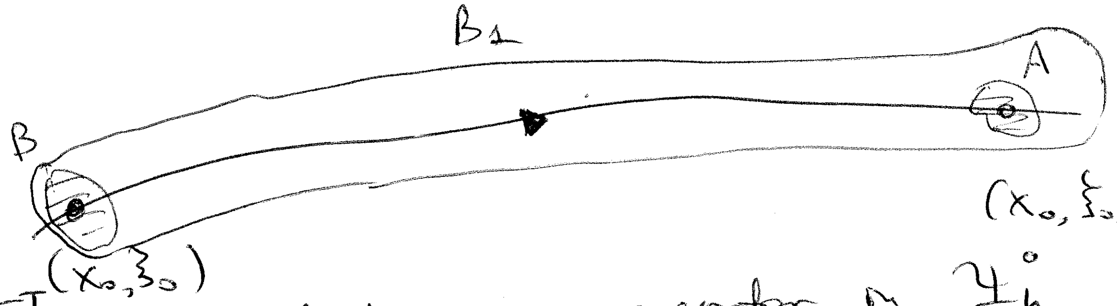
(here $Pu = f$ as before, $P \in \Psi_h^k$)

Then $Au \in H^s$ and $\exists\, \chi \in C_c^\infty(U)$ s.t. for all N

$$\|Au\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} + C_N h^N \|\chi u\|_{H_h^{-N}}$$

<u>Remarks</u>: 1 Can reverse the direction of the flow (i.e. take $T \leq 0$)

2 · PoS III $\Rightarrow$ PoS II : take $u = u(h)$,
assume we have $\varphi_{-T}(x_0, \xi_0) \notin WF_h(u)$,
and $\varphi_{-t}(x_0, \xi_0) \notin WF_h(f)$ for $0 \le t \le T$.
Need to show $(x_0, \xi_0) \notin WF_h(u)$.
Arrange things as follows:



So that:
- A is any operator in Ψ_h^0 such that $WF_h(A) \subset$ small nbhd of (x_0, ξ_0)
- the control condition holds
- $WF_h(B) \subset$ small enough nbhd of $\varphi_{-T}(x_0, \xi_0)$ so that $Bu = O(h^\infty)_{C^\infty}$ (using here that $\varphi_{-T}(x_0, \xi_0) \notin WF_h(u)$)
- $WF_h(B_1) \subset$ small enough nbhd of $\varphi_{[-T,0]}(x_0, \xi_0)$ so that $B_1 f = O(h^\infty)_{C^\infty}$ (using here that $\varphi_{[-T,0]} \cap WF_h(f) = \emptyset$)

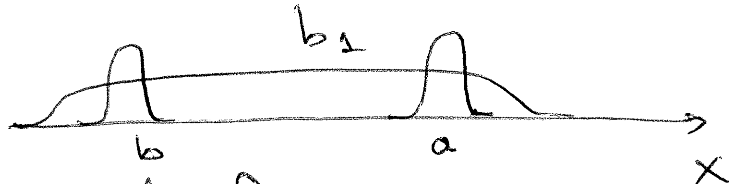
Applying PoS III we see that all the RHS terms are $O(h^\infty)$ for any s
$\Downarrow$
$Au = O(h^\infty)_{C^\infty}$, so we get $(x_0, \xi_0) \notin WF_h(u)$.

(3) Let's come back to our basic example:

$P = hD_x$ (let's put $\nu = 0$ for simplicity)

$P = \frac{h}{i}\partial_x$. Let's take A, B, B_1 multiplication operators: $A \to a(x)$, $B \to b(x)$, $B_1 \to b_1(x)$.

The control condition just gives the picture



And we get for $s = 0$

in this simple case actually no remainder

$$\|au\|_{L^2} \le C\|bu\|_{L^2} + Ch^{-1}\|b_1 hu'\|_{L^2} + \cancel{O(h^\infty)}\|u\|_{H_h^{-N}}$$

I.e. $\|au\|_{L^2} \le C\|bu\|_{L^2} + C\|b_1 \cdot u'\|_{L^2}$.

This is a fancier version of the Fundamental Theorem of Calculus:

$$|u(1)| \le |u(0)| + C\sqrt{\int_0^1 |u'|^2}.$$

This explains the power h^{-1} in the estimate & the loss of regularity compared to the elliptic bound: need $\|f\|_{H_h^{s-k+1}}$ instead of $\|f\|_{H_h^{s-k}}$.

This is similar to loss of derivatives for the wave equation $(-\partial_t^2 + \Delta_x)u = f$: we have

$$\|u(t,\cdot)\|_{H^1_x} \le \|u(0,\cdot)\|_{H^1_x} + C\|f\|_{L^2_t[0,T]L^2_x}$$

So $k = 2$ but the L^2 norm of f gives the H^1 norm of u, not H^2...