Properties of semiclassical quantization

Recall: $a(x,\xi;h) \in S^k_{1,0,h}$ if

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_{\alpha\beta} \langle \xi \rangle^{k-|\beta|}, \quad (x,\xi)\in\mathbb{R}^{2n},\ 0<h\le 1.$$

$$Op_h(a) f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) f(y)\, dy\, d\xi,$$

$$Op_h(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

Oscillatory testing

READ [Zw, Theorem 4.19]

Want to recover the symbol a from the operator $Op_h(a)$.

1) Assume $a \in S^k_{h,1,0}$. Denote for $\eta \in \mathbb{R}^n$, the function $e_\eta(x) = e^{\frac{i}{h}\langle x,\eta\rangle}$. Then

$$\boxed{Op_h(a)\, e_\eta(x) = e^{\frac{i}{h}\langle x,\eta\rangle} \cdot a(x,\eta;h)}$$

Here we can apply $Op_h(a)$ to e_η since $e_\eta \in \mathcal{S}'$.

Proof. Will just do the case $a \in C_c^\infty(\mathbb{R}^n)$. Then

$$Op_h(a) e_\eta(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi)\, \hat{e}_\eta\left(\frac{\xi}{h}\right) d\xi.$$

But $\hat{e}_\eta(\xi) = (2\pi)^n \delta\left(\xi - \frac{\eta}{h}\right)$, so $\hat{e}_\eta\left(\frac{\xi}{h}\right) = (2\pi h)^n \delta(\xi-\eta)$ finishing the proof. $\square$

2) Assume $A : \mathcal{S}' \to \mathcal{S}'$ is continuous and $a \in S^k_{1,0,h}$ satisfy

$$\boxed{A e_\eta(x) = e^{\frac{i}{h}\langle x,\eta\rangle} a(x,\eta;h) \quad \forall \eta}$$

Then $A = Op_h(a)$

Proof. ~~Put~~ Take $u \in \mathcal{S}(\mathbb{R}^n)$ & write
$$u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} \hat{u}\left(\tfrac{\eta}{h}\right) e_\eta(x)\, d\eta$$
by the Fourier inversion formula. Since A is continuous $\mathcal{S}' \to \mathcal{S}'$, $\|e_\eta\|_{\mathcal{S}'}$ is polynomially bdd in η for any seminorm in $\mathcal{S}'$ (with the polynomial dependency on the seminorm chosen), and $\hat{u}$ is rapidly decaying, we get
$$Au(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} \hat{u}(\eta/h)\, Ae_\eta(x)\, d\eta$$
$$= (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x, \eta\rangle} a(x,\eta;h)\, \hat{u}(\eta/h)\, d\eta = \mathrm{Op}_h(a)u.$$
So $A = \mathrm{Op}_h(a)$ on $\mathcal{S} \Rightarrow A = \mathrm{Op}_h(a)$ on $\mathcal{S}'$ by density.

Product Rule

$\boxed{\text{READ [Zw, Thm 4.11 + 4.17]}}$

Assume $a \in S^k_{1,0,h}$, $b \in S^\ell_{1,0,h}$. Then $\exists\, a\#b \in S^{k+\ell}_{1,0,h}$ such that $\mathrm{Op}_h(a)\mathrm{Op}_h(b) = \mathrm{Op}_h(a\#b)$ and moreover
$$a\#b(x,\xi;h) \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi;h)\, \partial_x^\alpha b(x,\xi;h). \qquad \text{(PROD)}$$

Proof. Will just do the case $a, b \in C_c^\infty(\mathbb{R}^{2n})$, h-indepdt.
We use oscillatory testing. Define $\forall \eta$,
$$a\#b(x,\eta;h) = e^{-\frac{i}{h}\langle x,\eta\rangle}\, \mathrm{Op}_h(a)\mathrm{Op}_h(b) e_\eta(x).$$
Note: $\forall h$, $a\#b \in C^\infty(\mathbb{R}^{2n})$ because $\mathrm{Op}_h(a), \mathrm{Op}_h(b) : \mathcal{S}' \to \mathcal{S}$.
So if we show that $a\#b \in S^{k+\ell}_{1,0,h}$ & (PROD) holds then the proof is finished.

Using oscillatory testing, we write

$Op_h(b)e_\eta(x) = e_\eta(x) \cdot b(x,\eta)$. Therefore

$$a\# b(x,\eta;h) = e^{-\frac{i}{h}\langle x,\eta\rangle} Op_h(a)\left[e^{\frac{i}{h}\langle \cdot,\eta\rangle} b(\cdot,\eta)\right]$$

$$= (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{h}\langle x,\eta\rangle} \cdot e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi)\, e^{\frac{i}{h}\langle y,\eta\rangle} b(y,\eta)\,dy\,d\xi$$

$$= (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi-\eta\rangle} a(x,\xi)\, b(y,\eta)\,dy\,d\xi$$

Note: $a\# b \in C_c^\infty(\mathbb{R}^{2n})$ for any fixed h since $a, b \in C_c^\infty(\mathbb{R}^{2n})$, and the $\int$ is over a compact set.

For fixed x,η, can get an asymptotic expansion for $a\# b(x,\eta;h)$ as $h \to 0$ by <u>Method of Stationary Phase</u> (MSPh)

Here $\Phi(y,\xi) = \langle x-y, \xi-\eta\rangle$. So,

$\nabla\Phi = 0 \iff y = x,\ \xi = \eta$. And $|\det \nabla^2\Phi| = 1$, $\operatorname{sgn} \nabla^2\Phi = 0$, $\Phi(x,\eta) = 0$.

(note: $\Phi = -\langle y,\xi\rangle +$ linear terms $+$ cst terms in y,ξ).

So the first term in the expansion is

$$(2\pi h)^{-n} \cdot \left[(2\pi h)^n \cdot e^{\frac{i\pi}{4}\cdot 0} \cdot 1^{-1/2} \cdot e^{\frac{i}{h}\cdot 0} \cdot a(x,\eta)\, b(x,\eta)\right].$$

For next terms, need to look at the proof of stationary phase - since Φ is quadratic, there is an explicit expansion & it gives (PROD).

Finally, MSPh actually gives an expansion uniform in (x,η) with all (x,η)-derivatives. $\square$

Basic consequences of (PROD):

1. $a \# b = ab + hS^{k+l-1}_{1,0,h}$.

2. The Commutator Rule: $[Op_h(a), Op_h(b)] = Op_h(a\#b - b\#a)$ &

$$a\#b - b\#a = -ih\{a,b\} + h^2 S^{k+l-2}_{1,0,h}$$

where $\{a,b\}(x,\xi) = \sum_{j=1}^{n} [\partial_{\xi_j} a(x,\xi)\partial_{x_j} b(x,\xi) - \partial_{x_j} a(x,\xi)\partial_{\xi_j} b(x,\xi)]$

is the Poisson bracket.

3. Pseudolocality: Assume $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$. Then $\forall a \in S^k_{1,0,h}$,

$\chi_1 Op_h(a) \chi_2 = O(h^\infty)_{\mathcal{D}' \to C_c^\infty}$, i.e. it has an integral kernel in $h^\infty \cdot C_c^\infty(\mathbb{R}^{2n})$.

Proof. $\chi_j = Op_h(\chi_j(x))$. Use the Product Rule.

Get $\chi_1 Op_h(a)\chi_2 = Op_h(c)$ where $c = \chi_1 \# a \# \chi_2$.

But from (PROD) we see that all terms in the expansion are 0, since $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$. So $c \in h^\infty S^{-\infty}$ which implies that the integral kernel of $Op_h(c)$ is in $h^\infty C^\infty$: $Op_h(c)f(x) = \int_{\mathbb{R}^n} K(x,y;h) f(y)\,dy$ where $K \in C^\infty$ & $\sup |\partial_x^\alpha \partial_y^\beta K| \le C_{\alpha\beta N} h^N$ $\forall \alpha,\beta,N$.

Finally $K \in C_c^\infty$ since $\operatorname{supp} K \subset \operatorname{supp}\chi_1 \times \operatorname{supp}\chi_2$. $\square$

Adjoint Rule: Assume $a \in S^k_{1,0,h}$. Then $\exists\, a^* \in S^k_{1,0,h}$ s.t. $Op_h(a)^* = Op_h(a^*)$ and

$$a^*(x,\xi;h) \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi;h)}. \quad \text{(ADJ)}$$

Note: here $Op_h(a)^* = Op_h(a^*)$ just means that

$$\forall f,g \in C_c^\infty(\mathbb{R}^n),\ \langle Op_h(a)f, g\rangle_{L^2} = \langle f, Op_h(a^*)g\rangle_{L^2}.$$

Proof. We have for $g \in C_c^\infty(\mathbb{R}^n)$,

$$Op_h(a)^* g(y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle y-x,\xi\rangle} \overline{a(x,\xi;h)} g(x)\, d\xi\, dx$$

i.e. $$Op_h(a)^* g(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} \overline{a(y,\xi;h)}\, g(y)\, dy\, d\xi.$$

Use oscillatory testing $\to$ set

$$a^*(x,\eta;h) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi-\eta\rangle} \overline{a(y,\xi;h)}\, dy\, d\xi.$$

Now use Method of Stationary Phase similarly to Product Rule.
(even when $a \in C_c^\infty$, need not have $a^* \in C_c^\infty$ But it will be $h^\infty \langle\xi\rangle^{-\infty}$ away from supp a ...) $\square$

Basic consequence of (ADJ): $a^* = \bar{a} + h S^{k-1}_{1,0,h}$.

Mapping Properties

1 If $a \in S^0_{1,0,h}$, then $Op_h(a): \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ restricts to a bdd operator

$Op_h(a): L^2 \to L^2$ &

$\|Op_h(a)\|_{L^2 \to L^2}$ is bdd uniformly in h...

Proof in the case $a \in C_c^\infty(\mathbb{R}^{2n})$:

$Op_h(a)$ has integral kernel

$$K(x,y;h) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi)\, d\xi.$$

So if $\mathcal{F}_\xi$ denotes the Fourier tr. in ξ only, then

$$K(x,y;h) = (2\pi h)^{-n} \mathcal{F}_\xi a\left(x, \frac{y-x}{h}\right).$$

However, $\mathcal{F}_\xi a$ is Schwartz since $a \in C_c^\infty$.

Use Schur's Inequality: enough to estimate

$$\sup_x \int |K(x,y;h)|\,dy \le C$$

$$\sup_y \int |K(x,y;h)|\,dx \le C$$

by change of variables...

For general $a \in S^0_{1,0,h}$ this is harder.

See [Zw, Theorem 4.23]. $\square$

Sobolev spaces.

Recall $\langle \xi \rangle := \sqrt{1+|\xi|^2}$.

Usual Sobolev space: $H^k(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

$\cancel{\|u\|^2_{H^k(\mathbb{R}^n)} = \int \langle \xi \rangle}$

$$\|u\|_{H^k(\mathbb{R}^n)} := \| \langle \xi \rangle^k \hat{u}(\xi) \|_{L^2}.$$

Semiclassical Sobolev space: $H^k_h(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

Same as H^k as a space but the norm is h-dependent: $\|u\|_{H^k_h(\mathbb{R}^n)} := \| \langle h\xi \rangle^k \hat{u}(\xi) \|_{L^2}$.

~~Say~~ $\cancel{\|u\|^2_{H_h} = \|u\|^2_{L^2} +}$

For example, $\|u\|^2_{H^2_h} \sim \|u\|^2_{L^2} + \|h^2 \Delta u\|^2_{L^2}$.

Indeed, $\text{LHS}^2 = \int_{\mathbb{R}^n} (1+|h\xi|^2)^2 |\hat{u}(\xi)|^2 \, d\xi$

$\text{RHS}^2 = \int_{\mathbb{R}^n} (1+|h\xi|^4) |\hat{u}(\xi)|^2 \, d\xi$.

Or, $\|u\|^2_{H^1_h} = \|u\|^2_{L^2} + \|h\nabla u\|^2_{L^2}$.

Mapping Property 2: If $a \in S^k_{1,0,h}$, $s \in \mathbb{R}$, then $\|\mathrm{Op}_h(a)\|_{H^s_h \to H^{s-k}_h} \le C$ as $h \to 0$.

Proof: Note that

$$\|u\|^2_{H^s_h} = \|\langle hD_x\rangle^s u\|_{L^2} = \|Op_h(\langle\xi\rangle^s)u\|_{L^2}.$$

(check it!) And $Op_h(\langle\xi\rangle^s)Op_h(\langle\xi\rangle^t) = Op_h(\langle\xi\rangle^{s+t})$.

Then for $f \in S(\mathbb{R}^n)$,

$$\|Op_h(a)f\|_{H^{s-k}_h} = \|Op_h(\langle\xi\rangle^{s-k})Op_h(a)Op_h(\langle\xi\rangle^{-s})Op_h(\langle\xi\rangle^s)u\|_{L^2}$$

$$= \|Op_h(\langle\xi\rangle^{s-k}\# a \# \langle\xi\rangle^{-s})Op_h(\langle\xi\rangle^s)u\|_{L^2}$$

$$\le C\|Op_h(\langle\xi\rangle^s)u\|_{L^2} = C\|u\|_{H^s_h}$$

since $\langle\xi\rangle^{s-k}\# a \# \langle\xi\rangle^{-s} \in S^0_{1,0,h}$ by Product Rule & we used L^2 boundedness.

Ellipticity

We start with a global statement:

Thm Assume $a \in S^k_{1,0,h}$ & $\exists c > 0$: $|a(x,\xi;h)| \ge c\langle\xi\rangle^k$.

Then $\exists\, b \in S^{-k}_{1,0,h}$ s.t. $a\#b - 1,\ b\#a - 1 \in h^\infty S^{-\infty}$.

In particular, $Op_h(a)Op_h(b),\ Op_h(b)Op_h(a) = I + h^\infty$. smoothing.

Proof. 1) $a^{-1} \in S^{-k}_{1,0,h}$: direct verification.

So $Op_h(a)Op_h(a^{-1}) = Op_h(1+r)$, $r \in hS^{-1}_{1,0,h}$.

Need to invert $1+r$ → take the Neumann series: namely take $b \sim \sum_{j=0}^{\infty} a^{-1}\# r^j$, here $r^j \in h^j S^{-j}_{1,0,h}$. Then $a\#b - 1 \in h^\infty S^{-\infty}$

2) Take $\tilde{b}$ s.t. $\tilde{b}\#a - 1 \in h^\infty S^{-\infty}$. Then

$\tilde{b} + h^\infty S^{-\infty} = \tilde{b}\#a\#b = b + h^\infty S^{-\infty}$ ... ∎