18.155, FALL 2021, PROBLEM SET 9

Review / helpful information:

- Riemannian metric on an open subset $U \subset \mathbb{R}^n$: $g = \sum_{j,k=1}^n g_{jk}(x)\, dx_j dx_k$ where $g_{jk} \in C^\infty(U)$ and the matrix $G(x) = (g_{jk}(x))_{j,k=1}^n$ is symmetric and positive definite for all x. This gives an x-dependent inner product $g(x)$ on $\mathbb{R}^n = T_x U$ by the formula

$$\langle v, w\rangle_{g(x)} = \sum_{j,k=1}^n g_{jk}(x) v_j w_k \quad \text{for all} \quad v, w \in \mathbb{R}^n.$$

- For such a Riemannian metric, the volume measure is

$$d\,\mathrm{vol}_g(x) := \sqrt{\det G(x)}\, dx, \tag{1}$$

i.e. for each measurable $f : U \to \mathbb{C}$ we have

$$\int_U f(x)\, d\,\mathrm{vol}_g(x) = \int_U f(x) \sqrt{\det G(x)}\, dx.$$

- Riemannian metric g on a manifold M: an inner product $\langle \bullet, \bullet\rangle_{g(x)}$ on each tangent space $T_x M$, $x \in M$, which is smooth in x. The latter means that for each coordinate system $\varkappa : U_0 \to V_0$, where $U_0 \subset M$, $V_0 \subset \mathbb{R}^n$ are open, there exists a smooth Riemannian metric $\varkappa^{-*}g$ on V_0 (called the *pullback* of g by the parametrization $\varkappa^{-1}$) such that

$$\langle v, w\rangle_{g(x)} = \langle d\varkappa(x) v, d\varkappa(x) w\rangle_{\varkappa^{-*}g(\varkappa(x))} \quad \text{for all} \quad x \in M,\ v, w \in T_x M,$$

where we recall that $d\varkappa(x)v, d\varkappa(x)w \in \mathbb{R}^n$. The volume measure $d\,\mathrm{vol}_g$ on M is defined as follows: for each measurable $f : M \to \mathbb{C}$ supported inside the domain U_0 of some coordinate system $\varkappa : U_0 \to V_0$, we have

$$\int_M f(x)\, d\,\mathrm{vol}_g(x) = \int_{V_0} f(\varkappa^{-1}(y))\, d\,\mathrm{vol}_{\varkappa^{-*}g}(y) \tag{2}$$

where $d\,\mathrm{vol}_{\varkappa^{-*}g}$ is the volume measure of the metric $\varkappa^{-*}g$ on V_0, defined by (1). Exercise 1(a) below implies that this does not depend on the choice of coordinates.

- A diffeomorphism $\Phi : M \to \widetilde{M}$ of manifolds $M, \widetilde{M}$ with some given Riemannian metrics $g, \tilde{g}$ is called an *isometry* if

$$\langle d\Phi(x) v, d\Phi(x) w\rangle_{\tilde{g}(\Phi(x))} = \langle v, w\rangle_{g(x)} \quad \text{for all} \quad x \in M,\ v, w \in T_x M,$$

where we recall that $d\Phi(x)v, d\Phi(x)w \in T_{\Phi(x)}\widetilde{M}$.

- Distributions on a manifold M with a fixed Riemannian metric: $\mathcal{D}'(M)$ is the space of continuous linear functionals on $C_c^\infty(M)$. Embed $L^1_{\mathrm{loc}}(M)$ into $\mathcal{D}'(M)$ by the pairing

$$(f,\varphi)=\int_M f(x)\varphi(x)\,d\operatorname{vol}_g(x),\quad f\in L^1_{\mathrm{loc}}(M),\quad \varphi\in C_c^\infty(M).$$

- Basic properties of the pullback operators defined in Exercise 1(c) below:
  - if $\Phi: M_1\to M_2$ and $\Phi': M_2\to M_3$ are diffeomorphisms, then $(\Phi'\circ\Phi)^*=\Phi^*(\Phi')^*$;
  - if $\Phi: M\to\widetilde{M}$ is a diffeomorphism and $a\in C^\infty(\widetilde{M})$, $u\in\mathcal{D}'(\widetilde{M})$, then $\Phi^*(au)=(\Phi^*a)(\Phi^*u)$;
  - $\operatorname{supp}(\Phi^*u)=\Phi^{-1}(\operatorname{supp} u)$.
- If $U\subset\mathbb{R}^n$ is an open set and $P\in\operatorname{Diff}^m(U)$ is a differential operator, then for each $\varphi\in C^\infty(U;\mathbb{R})$ and $a\in C^\infty(U;\mathbb{C})$ we have

$$P(e^{i\lambda\varphi(x)}a(x))=e^{i\lambda\varphi(x)}\big(\sigma_m(P)(x,d\varphi(x))a(x)\lambda^m+\mathcal{O}(\lambda^{m-1})_{C^\infty(U)}\big)\quad\text{as}\quad\lambda\to\infty \tag{3}$$

  where $\sigma_m(P)\in C^\infty(U\times\mathbb{R}^n;\mathbb{R})$ is the principal symbol of P.

1. (a) Assume that $U,\widetilde{U}\subset\mathbb{R}^n$ are open sets, $\Phi: U\to\widetilde{U}$ is a diffeomorphism, and $g,\tilde g$ are some Riemannian metrics on $U,\widetilde{U}$, with the corresponding volume measures (defined in (1)) denoted $d\operatorname{vol}_g, d\operatorname{vol}_{\tilde g}$. Show that for each $\tilde f\in L^1_c(\widetilde{U})$ we have the change of variables formula

$$\int_{\widetilde{U}}\tilde f(y)\,d\operatorname{vol}_{\tilde g}(y)=\int_U \tilde f(\Phi(x))J_{\Phi,g,\tilde g}(x)\,d\operatorname{vol}_g(x)$$

for a certain positive function $J_{\Phi,g,\tilde g}\in C^\infty(U)$ (independent of $\tilde f$). Show furthermore than if $\Phi:(U,g)\to(\widetilde{U},\tilde g)$ is an isometry, then $J_{\Phi,g,\tilde g}=1$.

(b) (Optional) Let $\Phi: M\to\widetilde{M}$ be a diffeomorphism where $M,\widetilde{M}$ are manifolds, and fix Riemannian metrics $g,\tilde g$ on $M,\widetilde{M}$. Show that for each $\tilde f\in L^1_c(\widetilde{M})$ we have the change of variables formula

$$\int_{\widetilde{M}}\tilde f(y)\,d\operatorname{vol}_{\tilde g}(y)=\int_M f(\Phi(x))J_{\Phi,g,\tilde g}(x)\,d\operatorname{vol}_g(x) \tag{4}$$

for a certain positive function $J_{\Phi,g,\tilde g}\in C^\infty(M)$ (independent of $\tilde f$).

(c) Using part (b), show that the pullback operator $\Phi^*: L^1_{\mathrm{loc}}(\widetilde{M})\to L^1_{\mathrm{loc}}(M)$, $\Phi^*f:=f\circ\Phi$, extends to a sequentially continuous operator $\Phi^*:\mathcal{D}'(\widetilde{M})\to\mathcal{D}'(M)$.

2. This exercise discusses Sobolev spaces on a compact manifold M. We define a *cutoff atlas* to be a finite collection of coordinate systems $\varkappa_j: U_j\to V_j$ on M, $j=1,\dots,N$, such that $M=\bigcup_{j=1}^N U_j$, together with a partition of unity $\chi_j\in C_c^\infty(U_j)$, $\sum_{j=1}^N\chi_j=1$ on M. Let $s\in\mathbb{R}$ and fix a Riemannian metric g on M.

Fix a cutoff atlas and define the space $H^s(M) \subset \mathcal{D}'(M)$ as follows: a distribution u lies in $H^s(M)$ if and only if for each j the distribution $\varkappa_j^{-*}(\chi_j u)$ lies in $H^s(\mathbb{R}^n)$. Here $\varkappa_j^{-*}(\chi_j u) \in \mathcal{E}'(V_j)$ is the pullback of $\chi_j u \in \mathcal{E}'(U_j)$ by the diffeomorphism $\varkappa_j^{-1} : V_j \to U_j$, extended by 0 to an element of $\mathcal{E}'(\mathbb{R}^n)$. For $u \in H^s(M)$, define its norm $\|u\|_{H^s(M)}$ by

$$\|u\|_{H^s(M)}^2 := \sum_{j=1}^{N} \|\varkappa_j^{-*}(\chi_j u)\|_{H^s(\mathbb{R}^n)}^2.$$

(a) (Optional) Show that $H^s(M)$ is a Hilbert space. (Hint: to show completeness, assume $u^{(k)}$ is a Cauchy sequence in $H^s(M)$. First use completeness of $H^s(\mathbb{R}^n)$ to show that for each j, the sequence $\varkappa_j^{-*}(\chi_j u^{(k)})$ converges to some v_j in $H^s(\mathbb{R}^n)$. Next, show that v_j satisfy the compatibility conditions $(\varkappa_\ell \circ \varkappa_j^{-1})^*((\varkappa_\ell^{-*}\chi_j)v_\ell) = (\varkappa_j^{-*}\chi_\ell)v_j$ for all j, ℓ. Now, motivated by the identity

$$w = \sum_{\ell=1}^{N} \varkappa_\ell^*(\varkappa_\ell^{-*}(\chi_\ell w)) \quad \text{for all} \quad w \in \mathcal{D}'(M) \tag{5}$$

define $u := \sum_{\ell=1}^{N} \varkappa_\ell^* v_\ell$ and show that $u \in H^s(M)$ (which uses invariance of Sobolev classes under multiplications by smooth functions and under pullbacks) and $u^{(k)} \to u$ in $H^s(M)$ (which uses the compatibility conditions).)

(b) Show that if we take a different cutoff atlas on M, then the space $H^s(M)$ stays the same and the norms on $H^s(M)$ given by two different cutoff atlases are equivalent. (Hint: use the identity (5) and the fact that multiplications by smooth functions and pullbacks by diffeomorphisms define continuous operators on appropriate Sobolev spaces.)

3. (Optional) Let $\mathbb{S}^n = \{\theta \in \mathbb{R}^{n+1} : |\theta| = 1\}$ be the n-sphere, with $n \geq 2$. We endow it with the metric g which is the restriction of the Euclidean metric. In this exercise you compute the eigenvalues of the operator $-\Delta_g$, namely the numbers $\lambda \in \mathbb{R}$ such that there exist nonzero $u \in C^\infty(\mathbb{S}^n; \mathbb{R})$ solving the eigenfunction equation

$$-\Delta_g u = \lambda u.$$

(a) Show that each eigenvalue λ has to satisfy $\lambda \geq 0$. (Hint: compute the integral $\int_{\mathbb{S}^n} (\Delta_g u) u \, d\,\mathrm{vol}_g$ using the defining property of the Laplace–Beltrami operator.)

(b) Let $a \geq 0$. Denote by Δ_0 the usual Laplace operator on $\mathbb{R}^{n+1}$. Show that the equation

$$\Delta_0 v = 0 \quad \text{on} \quad \mathbb{R}^{n+1} \setminus \{0\} \tag{6}$$

has a nonzero solution $v \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ which is homogeneous of degree a if and only if a is a (nonnegative) integer. (Hint: show that v is a locally integrable function on $\mathbb{R}^{n+1}$ and defines a tempered distribution in $\mathscr{S}'(\mathbb{R}^{n+1})$, which we denote $\tilde{v}$. Arguing similarly to the proof of the theorem in §10.3 in lecture notes, show that $\Delta_0 \tilde{v} = 0$.

Now pass to the Fourier transform of $\tilde{v}$ and show that it is supported at a single point; deduce from here that $\tilde{v}$ is a polynomial.)

(c) The pullback of the operator Δ_0 by the polar coordinate diffeomorphism

$$\Phi : (0,\infty)\times\mathbb{S}^n \to \mathbb{R}^{n+1}\setminus\{0\},\quad \Phi(r,\theta) := r\theta$$

is equal to the operator $\partial_r^2 + \frac{n}{r}\partial_r + \frac{1}{r^2}\Delta_g$, with the spherical Laplacian Δ_g acting in the θ variable. (This can be checked by noting that this operator has to be the Laplace–Beltrami operator of the pullback by Φ of the Euclidean metric, but you don't need to do this computation here.) Using this, show that the eigenvalues of $-\Delta_g$ are given by $k(k+n-1)$ where k runs over nonnegative integers. (Hint: if u is an eigenfunction of $-\Delta_g$ then define $v(r\theta) = r^a u(\theta)$ in polar coordinates for a right choice of a so that $\Delta_0 v = 0$.) The eigenfunctions of $-\Delta_g$ are called *spherical harmonics*.

4. Let M be a manifold, $U_0 \subset M$ be an open set, and $\varkappa : U_0 \to V_0$, $\widetilde{\varkappa} : U_0 \to \widetilde{V}_0$ be two coordinate systems, where $V_0, \widetilde{V}_0 \subset \mathbb{R}^n$ are open. Assume that $P \in \mathrm{Diff}^m(M)$ is a differential operator. Denote by $\varkappa^{-*}P$, $\widetilde{\varkappa}^{-*}P$ the pullbacks of P by $\varkappa^{-1}$ and $\widetilde{\varkappa}^{-1}$, which are differential operators on V_0 and $\widetilde{V}_0$ respectively. Show that for each $x \in U_0$, $\xi \in T_x^*M$ we have the equality of principal symbols

$$\sigma_m(\varkappa^{-*}P)(\varkappa(x), d\varkappa(x)^{-T}\xi) = \sigma_m(\widetilde{\varkappa}^{-*}P)(\widetilde{\varkappa}(x), d\widetilde{\varkappa}(x)^{-T}\xi).$$

(Hint: use the pullback theorem from §14.1 in lecture notes. This implies that the principal symbol is invariantly defined as a function on the cotangent bundle.)

5. Let $U \subset \mathbb{R}^n$ be an open set. Show the following properties of principal symbols of operators on U. (All of the above properties are also satisfied on manifolds, which can be deduced from the case of open subsets of $\mathbb{R}^n$.)

(a) Product Rule: if $P \in \mathrm{Diff}^m(U)$, $Q \in \mathrm{Diff}^\ell(U)$, then $\sigma_{m+\ell}(PQ) = \sigma_m(P)\sigma_\ell(Q)$, where $PQ \in \mathrm{Diff}^{m+\ell}(U)$ is the composition of P and Q. (Hint: one way is to use (3).)

(b) Adjoint Rule: if $P \in \mathrm{Diff}^m(U)$ and we fix a Riemannian metric g on U, then there exists an adjoint operator $P^* \in \mathrm{Diff}^m(U)$ such that for all $\varphi \in C_c^\infty(M)$, $\psi \in C^\infty(M)$

$$\langle P\varphi, \psi\rangle_{L^2(U;d\,\mathrm{vol}_g)} = \langle \varphi, P^*\psi\rangle_{L^2(U;d\,\mathrm{vol}_g)},\quad \langle \varphi,\psi\rangle_{L^2(U;d\,\mathrm{vol}_g)} := \int_U \varphi(x)\overline{\psi(x)}\,d\,\mathrm{vol}_g(x)$$

and $\sigma_m(P^*) = \overline{\sigma_m(P)}$. (Hint: you will likely need to integrate by parts.)

(c) (Optional) Commutator Rule: if $P \in \mathrm{Diff}^m(U)$, $Q \in \mathrm{Diff}^\ell(U)$ then the commutator $[P,Q] := PQ - QP$ lies in $\mathrm{Diff}^{m+\ell-1}(U)$ and $\sigma_{m+\ell-1}([P,Q]) = -i\{\sigma_m(P), \sigma_\ell(Q)\}$ where the Poisson bracket $\{p,q\}$ of $p,q \in C^\infty(U\times\mathbb{R}^n)$ is defined by

$$\{p,q\} := \sum_{j=1}^n (\partial_{\xi_j}p)(\partial_{x_j}q) - (\partial_{x_j}p)(\partial_{\xi_j}q).$$