18.155, FALL 2021, PROBLEM SET 6

Review / helpful information:

- If $\Phi : U \to V$ is a C^∞ submersion, then $\Phi^* : \mathcal{D}'(V) \to \mathcal{D}'(U)$ is the unique sequentially continuous operator such that $\Phi^* f = f \circ \Phi$ for all $f \in L^1_{\mathrm{loc}}(V)$.
- You may use without proof the following corollary of the Inverse Mapping Theorem: if Φ is a submersion, then for each open set $\widetilde{U} \subset U$, the set $\Phi(\widetilde{U})$ is open.
- Advanced fundamental solution $E \in \mathcal{D}'(\mathbb{R}^4)$ of the wave operator $\Box = \partial_{x_0}^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2$:

$$(E, \varphi) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varphi(|x'|, x')}{|x'|} \, dx' \quad \text{for all} \quad \varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^4). \tag{1}$$

1. (Optional) Let $\Phi : U \to V$ be a submersion and $v \in \mathcal{D}'(V)$.

(a) Assume that $\widetilde{U} \subset U$, $\widetilde{V} \subset V$ are open sets such that $\Phi(\widetilde{U}) \subset \widetilde{V}$ and thus $\widetilde{\Phi} := \Phi|_{\widetilde{U}}$ is a submersion from $\widetilde{U}$ to $\widetilde{V}$. Show that $(\Phi^* v)|_{\widetilde{U}} = \widetilde{\Phi}^*(v|_{\widetilde{V}})$.

(b) Show that if $\Phi(U) = V$, then $\Phi^* : \mathcal{D}'(V) \to \mathcal{D}'(U)$ is injective. (You might need to review the construction of Φ^* in Lecture 10.)

(c) Show that $\operatorname{supp}(\Phi^* v) = \Phi^{-1}(\operatorname{supp} v)$ and $\operatorname{sing\,supp}(\Phi^* v) \subset \Phi^{-1}(\operatorname{sing\,supp} v)$. (One actually has $\operatorname{sing\,supp}(\Phi^* v) = \Phi^{-1}(\operatorname{sing\,supp} v)$ but let's skip this one.)

2. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be given by $\Phi(x) = x^2$. Show that the pullback operator $\Phi^* : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ does not extend to a sequentially continuous operator $\mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$. (Hint: let $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ be equal to 1 near 0, put $\chi_\varepsilon(x) := \varepsilon^{-1}\chi(x/\varepsilon)$, and look at the limit of $(\Phi^* \chi_\varepsilon, \chi)$.)

3. If $\Phi : U \to V$ is a C^∞ map, then $\Phi^* : C^\infty(V) \to C^\infty(U)$ is well-defined. Denote by $(\Phi^*)^t : C_{\mathrm{c}}^\infty(U) \to \mathcal{E}'(V)$ the transpose of Φ^*, defined by

$$((\Phi^*)^t \varphi, \psi) = (\Phi^* \psi, \varphi) \quad \text{for all} \quad \varphi \in C_{\mathrm{c}}^\infty(U), \ \psi \in C^\infty(V).$$

Compute the transposes of the following two simple maps. In each case decide whether $(\Phi^*)^t$ maps $C_{\mathrm{c}}^\infty(U)$ to $C_{\mathrm{c}}^\infty(V)$ (which would allow to extend Φ^* to distributions):

(a) $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi(x_1, x_2) = x_1$;

(b) $\Phi : \mathbb{R} \to \mathbb{R}^2$, $\Phi(x_1) = (x_1, 0)$.

4. Assume that $W \subset \mathbb{R}^n$ is open and $F : W \to \mathbb{R}^m$ is a C^∞ map. Define the submersion $\Phi : W \times \mathbb{R}^m \to \mathbb{R}^m$ by $\Phi(x, y) = y - F(x)$.

(a) Show that for each $u \in \mathcal{D}'(\mathbb{R}^m)$ the distribution $\Phi^* u \in \mathcal{D}'(W \times \mathbb{R}^m)$ is given by

$$(\Phi^* u, \varphi) = \Big(u(y), \int_W \varphi(x, y + F(x))\, dx \Big) \quad \text{for all} \quad \varphi \in C_c^\infty(W \times \mathbb{R}^m). \tag{2}$$

(Hint: start with $u \in C^\infty(\mathbb{R}^m)$ and extend by density.)

(b) Show that the Schwartz kernel of the pullback operator $F^* : C^\infty(\mathbb{R}^m) \to C^\infty(W)$ is given by $Q(x,y) = \delta_0(y - F(x))$ where $\delta_0(y - F(x))$ is defined as $\Phi^*\delta_0$. (In the special case when F is the identity map we see that the Schwartz kernel of the identity operator is given by $\delta(y - x)$.)

5. (Optional) Check that the distribution E given in (1) satisfies $\Box E = \delta_0$ directly, without appealing to the classification of distributions supported at the origin. To do this, introduce the spherical coordinates $x' = r\theta$ where $\theta \in \mathbb{S}^2$. You may use the formula

$$\Delta_{x'} = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_\theta$$

where $\Delta_\theta : C^\infty(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2)$ is the Laplace–Beltrami operator for the standard metric on the 2-sphere. You may also use that $\Delta_\theta f$ integrates to 0 on $\mathbb{S}^2$ for all $f \in C^\infty(\mathbb{S}^2)$. After getting rid of Δ_θ, you might find it useful to write everything in terms of the function $\psi(u, v, \theta) = \varphi(u + v, (u - v)\theta)$ where $\varphi \in C_c^\infty(\mathbb{R}^4)$ and $u, v \in \mathbb{R}$, $\theta \in \mathbb{S}^2$.

6. Let $E \in \mathcal{D}'(\mathbb{R}^4)$ be defined in (1).

(a) Assume that $w \in \mathcal{D}'(\mathbb{R}^4)$ and $\operatorname{supp} w \subset \{x_0 \geq 0\}$. Show that for each $\varphi \in C_c^\infty(\mathbb{R}^4)$ we have

$$(E * w, \varphi) = (w, \psi)$$

for some $\psi \in C_c^\infty(\mathbb{R}^4)$ such that

$$\psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varphi(x_0 + |y'|, x' + y')}{|y'|}\, dy', \quad x_0 \geq 0.$$

(b) Using part (a) and the formulas from §10.3 in lecture notes, show the following version of Kirchhoff's formula: if $u \in C^2(\{x_0 \geq 0\})$ is the solution to

$$\Box u = 0, \quad u|_{x_0=0} = 0, \quad \partial_{x_0} u|_{x_0=0} = g_1(x'),$$

then we have for all $x_0 \geq 0$ and $x' \in \mathbb{R}^3$

$$u(x_0, x') = \frac{x_0}{4\pi} \int_{\mathbb{S}^2} g_1(x' + x_0\theta)\, dS(\theta).$$

That is, the value of the solution at time x_0 and space x' is equal to x_0 times the average of the initial data g_1 over the sphere of radius x_0 centered at x'.