18.155, FALL 2021, PROBLEM SET 5

Review / helpful information:

- Convolution of compactly supported distributions: if $u, v \in \mathcal{E}'(\mathbb{R}^n)$ then $u * v \in \mathcal{E}'(\mathbb{R}^n)$ is defined by
$$(u * v, \varphi) = (u(x) \otimes v(y), \varphi(x+y)) \quad \text{for all} \quad \varphi \in C^\infty(\mathbb{R}^n).$$
- Two closed sets $V_1, V_2 \subset \mathbb{R}^n$ *sum properly* if for each $R > 0$ there exists $T(R) > 0$ such that for all $x \in V_1$, $y \in V_2$ such that $|x+y| \le R$, we have $|x|, |y| \le T(R)$.
- If $u, v \in \mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{supp} u, \operatorname{supp} v$ sum properly, then define $u * v \in \mathcal{D}'(\mathbb{R}^n)$ by
$$(u * v, \varphi) = (u(x) \otimes v(y), \chi(x)\chi(y)\varphi(x+y))$$
for each $\varphi \in C_c^\infty(\mathbb{R}^n)$. Here $\chi \in C_c^\infty(\mathbb{R}^n)$ is chosen so that $\chi = 1$ near $\overline{B(0, T(R))}$ where $\operatorname{supp} \varphi \subset B(0, R)$. (The result does not depend on the choice of χ.) In other words,
$$u * v|_{B(0,R)} = (\chi u) * (\chi v)|_{B(0,R)} \quad \text{if} \quad \chi \in C_c^\infty(\mathbb{R}^n),\ \operatorname{supp}(1-\chi) \cap \overline{B(0,T(R))} = \emptyset.$$
We have $\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v$.
- $E \in \mathcal{D}'(\mathbb{R}^n)$ is a *fundamental solution* of a constant coefficient differential operator P, if $PE = \delta_0$. In this case, if $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{supp} u$, $\operatorname{supp} E$ sum properly, then $u = E * (Pu) = P(E * u)$.
- A fundamental solution for $\partial_{x_1}^2 - \partial_{x_2}^2$ on $\mathbb{R}^2$ is given by
$$E(x_1, x_2) = \begin{cases} \frac{1}{2}, & x_1 > |x_2|; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

1. (Optional) Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open and fix $Q \in C^\infty(U \times V)$. Let $A : C_c^\infty(V) \to \mathcal{D}'(U)$ be the operator with Schwartz kernel Q. Show that A extends to a sequentially continuous operator $\widetilde{A} : \mathcal{E}'(V) \to C^\infty(U)$. (Such operators are called *smoothing*, we will encounter them again later in the course. The converse is true, a version of the Schwartz kernel theorem.)

(Hint: for $v \in \mathcal{E}'(V)$, define $\widetilde{A}v(x) := (v(y), Q(x,y))$. The smoothness of this can be proved similarly to, or deduced from by using cutoffs, the lemma in §7.1 in lecture notes. For sequential continuity, if $v_k \to 0$ in $\mathcal{E}'(V)$, which automatically implies that $\operatorname{supp} v_k$ all lie in a fixed compact subset of V, you can use Banach–Steinhaus for distributions to see that every derivative of $\widetilde{A}v_k$ is bounded locally uniformly. On the other hand, each derivative of $\widetilde{A}v_k$ goes to 0 pointwise. Now you can use Arzelà–Ascoli.)

2. Assume that $\operatorname{Re} a, \operatorname{Re} b > 0$. Show that $x_+^{a-1} * x_+^{b-1} = B(a,b)x_+^{a+b-1}$ where B denotes the beta function. (You can use the standard integral formula for convolution, no need to do things distributionally here. Note: using analytic continuation one can show that the same formula actually holds for all $a, b \in \mathbb{C}$, but you don't have to do this.)

3. Denote elements in $\mathbb{R}^n$ (where $n \geq 2$) by $x = (x_1, x')$ where $x' \in \mathbb{R}^{n-1}$. Define the set $\Omega := \{x \colon x_1 \geq |x'|\}$. Show that $\Omega + \Omega = \Omega$. Show also that Ω sums properly with the set $\{x_1 \geq 0\}$. Does the set $\{x_1 \geq 0\}$ sum properly with itself?

4. (Optional) Show that a fundamental solution for the Cauchy–Riemann operator $P := \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ on $\mathbb{R}^2$ is given by the locally integrable function

$$E(x_1, x_2) = \frac{1}{\pi(x_1 + ix_2)}.$$

5. Using the fact that the Heaviside function is a fundamental solution for ∂_x, show that for $u \in \mathcal{D}'(\mathbb{R})$, if $\operatorname{supp} u \subset [a, \infty)$ and $\operatorname{supp}(\partial_x u) \subset [b, \infty)$ for some $a \leq b$, then $\operatorname{supp} u \subset [b, \infty)$. Could we remove the condition that $\operatorname{supp} u \subset [a, \infty)$?

6. This exercise studies solutions to the initial value problem for the wave operator on $\mathbb{R}^2$, $P := \partial_{x_1}^2 - \partial_{x_2}^2$. Assume that

$$Pu = f, \quad u(0, x_2) = g_0(x_2), \quad \partial_{x_1} u(0, x_2) = g_1(x_2)$$

Here $u \in C^2(\mathbb{R}^2)$ is the solution, $f \in C^0(\mathbb{R}^2)$ is the forcing term, and $g_0 \in C^2(\mathbb{R}), g_1 \in C^1(\mathbb{R})$ are the initial data.

(a) Define $v(x_1, x_2) = H(x_1)u(x_1, x_2) \in \mathcal{D}'(\mathbb{R}^2)$ where H is the Heaviside function. Show that, with derivatives in the sense of distributions,

$$Pv = \delta_0'(x_1) \otimes g_0(x_2) + \delta_0(x_1) \otimes g_1(x_2) + H(x_1)f.$$

(b) Using that $\operatorname{supp} v \subset \{x_1 \geq 0\}$ show that $v = E * (Pv)$ where E is defined in (1).

(c) Assume that $w \in \mathcal{D}'(\mathbb{R}^2)$ and $\operatorname{supp} w \subset \{x_1 \geq 0\}$. Show that for each $\varphi \in C_c^\infty(\mathbb{R}^2)$ we have

$$(E * w, \varphi) = (w, \psi)$$

for some $\psi \in C_c^\infty(\mathbb{R}^2)$ such that

$$\psi(x) = \frac{1}{2} \int_{|y_2| < y_1} \varphi(x + y)\, dy, \quad x_1 \geq 0.$$

(d) (Optional) Using parts (a)–(c), show d'Alembert's formula: for $x_1 > 0$

$$\begin{aligned} u(x_1, x_2) = &\frac{1}{2}(g_0(x_2 + x_1) + g_0(x_2 - x_1)) + \frac{1}{2}\int_{x_2 - x_1}^{x_2 + x_1} g_1(s)\, ds \\ &+ \frac{1}{2}\int_0^{x_1}\int_{x_2-(x_1-\tau)}^{x_2+(x_1-\tau)} f(\tau, s)\, ds d\tau. \end{aligned} \tag{2}$$

(This would need a fair amount of computation.)

(e) Assume that $f = 0$ and $\operatorname{supp} g_0, \operatorname{supp} g_1 \subset [-R, R]$. Show that

$$\operatorname{supp} u \cap \{x_1 > 0\} \;\subset\; \{|x_2| \leq x_1 + R\}.$$

(This is called 'finite speed of propagation'.)

(f) Assume that $g_0 = g_1 = 0$ and $\operatorname{supp} f \subset \{x_1 > 0\}$. Show that singularities propagate at unit speed: namely, if $x \in \operatorname{sing\,supp} u$ and $x_1 > 0$, then we have $x = y + (t, -t)$ or $x = y + (t, t)$ for some $t \geq 0$ and $y \in \operatorname{sing\,supp} f$. (Hint: what is $\operatorname{sing\,supp} E$?)