# 18.155, FALL 2021, PROBLEM SET 3

Review / helpful information:

- Support: $x \in U$ does not lie in $\operatorname{supp} u$, where $u \in \mathcal{D}'(U)$, iff there is a neighborhood $V$ of $x$ in $U$ such that $(u, \varphi) = 0$ for all $\varphi \in C_c^\infty(U)$ with $\operatorname{supp} \varphi \subset V$.
- For $u \in \mathcal{D}'(U)$, we have $u|_{U \setminus \operatorname{supp} u} = 0$.
- $\mathcal{E}'(U)$ consists of distributions in $\mathcal{D}'(U)$ which have compact support. We can define the pairing $(u, \varphi)$ for $u \in \mathcal{E}'(U)$ and $\varphi \in C^\infty(U)$.
- Convergence in $C^\infty(U)$: we say $\varphi_k \to \varphi$ iff for each compact $K \subset U$ and each multiindex $\alpha$ we have $\sup_K |\partial^\alpha(\varphi_k - \varphi)| \to 0$.
- Homogeneous distributions: $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree $a \in \mathbb{C}$ if $(u, \varphi) = t^a(u, \varphi_t)$ for all $t > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, where we put $\varphi_t(x) := t^n \varphi(tx)$.
- For $a \in \mathbb{C}$, $\operatorname{Re} a > -1$ define $x_+^a := x^a$ when $x > 0$ and $x_+^a = 0$ when $x < 0$. Similarly define $x_-^a := (-x)_+^a$. These distributions can be defined for all $a \in \mathbb{C} \setminus -\mathbb{N}$ using the identity $\partial_x(x_+)^a = a x_+^{a-1}$.
- Principal value distribution $1/x$ is defined as the distributional derivative of $\log |x|$.

**1.** Show the following basic properties of support of a distribution:

**(a)** If $u \in \mathcal{D}'(U)$ and $V \subset U$ is open, then $\operatorname{supp}(u|_V) = \operatorname{supp} u \cap V$.

**(b)** If $u \in \mathcal{E}'(U)$ and $V \supset U$ is open, then there exists unique $v \in \mathcal{E}'(V)$ such that $v|_U = u$ and $\operatorname{supp} v \subset \operatorname{supp} u$. (This is the distributional analogue of the extension by zero operator $C_c^\infty(U) \to C_c^\infty(V)$.)

**(c)** If $u \in \mathcal{D}'(U)$, then $\operatorname{supp} \partial_{x_j} u \subset \operatorname{supp} u$.

**(d)** If $u \in \mathcal{D}'(U)$ and $a \in C^\infty(U)$, then $\operatorname{supp}(au) \subset \operatorname{supp} u \cap \operatorname{supp} a$.

**(e)** If $u \in \mathcal{D}'(U)$ then $\operatorname{supp} u \cap \{x \in U \mid a(x) \neq 0\} \subset \operatorname{supp}(au)$. In particular, if $au = 0$ then $\operatorname{supp} u \subset \{x \in U \mid a(x) = 0\}$.

**2.** Let $U \subset \mathbb{R}^n$ be open. This exercise elaborates on the metric topology on $C^\infty(U)$.

**(a)** Take a sequence of compact subsets $K_1 \subset K_2 \subset \dots$ of $U$ such that $U = \bigcup_N K_N^\circ$. Define the seminorms on $C^\infty(U)$

$$\|\varphi\|_N := \max_{|\alpha| \le N} \sup_{K_N} |\partial^\alpha \varphi|.$$

Show that $\varphi_k \to \varphi$ in $C^\infty(U)$ (according to the definition given in the beginning of this problemset) if and only if $\|\varphi_k - \varphi\|_N \to 0$ for all $N$.

**(b)** Show that the space $C^\infty(U)$ with the seminorms $\|\bullet\|_N$ is complete in the following sense: if $\varphi_k \in C^\infty(U)$ is a sequence such that $\sup_{j,k\geq r}\|\varphi_j - \varphi_k\|_N \to 0$ as $r\to\infty$ for all $N$, then there exists $\varphi \in C^\infty(U)$ such that $\varphi_k \to \varphi$ in $C^\infty(U)$. (A vector space with a countable family of seminorms which make it complete in the above sense is called a *Fréchet space.*)

**(c)** For $\varphi, \psi \in C^\infty(U)$, define

$$d(\varphi,\psi) := \sum_{N=1}^{\infty} 2^{-N}\frac{\|\varphi-\psi\|_N}{1+\|\varphi-\psi\|_N}.$$

Show that $d$ defines a metric on $C^\infty(U)$.

**(d)** Show that $\varphi_k \to \varphi$ in $C^\infty(U)$ if and only if $d(\varphi_k,\varphi)\to 0$ as $k\to\infty$.

**(e)** Show that the space $C^\infty(U)$ is complete with the metric $d(\bullet,\bullet)$.

**3.** Show that if $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree $a$, then $x_j u$ is homogeneous of degree $a+1$ and $\partial_{x_j} u$ is homogeneous of degree $a-1$. What is the degree of homogeneity of $\partial^\alpha \delta_0$?

**4.** (Optional) This exercise explores homogeneous distributions on $\mathbb{R}$ which are an alternative to $x_+^a$ and $x_-^a := (-x)_+^a$.

**(a)** For $\varepsilon > 0$ and $a \in \mathbb{C}$, define $(x+i\varepsilon)^a \in C^\infty(\mathbb{R})$ by the formula $(x+i\varepsilon)^a := \exp(a\log(x+i\varepsilon))$ where we use the branch of log on $\mathbb{C}\setminus(-\infty,0]$ which sends $(0,\infty)$ to reals. Similarly we can define $(x-i\varepsilon)^a$. Show that there exist limits in $\mathcal{D}'(\mathbb{R})$

$$(x\pm i0)^a = \lim_{\varepsilon\to 0+}(x\pm i\varepsilon)^a \ \in\ \mathcal{D}'(\mathbb{R}).$$

(Hint: for $\operatorname{Re} a > -1$ this is direct and $(x\pm i0)^a$ are locally integrable functions. For $a=-1$, write $(x\pm i\varepsilon)^{-1} = \partial_x \log(x\pm i\varepsilon)$ and note that $\log(x\pm i\varepsilon)$ has a distributional limit which is in $L^1_{\mathrm{loc}}(\mathbb{R})$. For general $a\neq -1$, reduce to the case of $a+1$ by antidifferentiation, similarly to what was done for $x_+^a$ in lecture.)

**(b)** For $a \in \mathbb{C}\setminus -\mathbb{N}$, express $(x\pm i0)^a$ as a linear combination of $x_+^a$ and $x_-^a$. (Hint: it is enough to consider the case $\operatorname{Re} a > -1$ by analytic continuation.)

**(c)** Show that $(x-i0)^{-1} - (x+i0)^{-1} = 2\pi i\delta_0$ and $(x+i0)^{-1} + (x-i0)^{-1}$ is twice the prinicipal value distribution $1/x$. (Hint: write $(x\pm i0)^{-1} = \partial_x \log(x\pm i0)$. Note that $\log(x\pm i0) = \log x$ for $x>0$ and $\log(x\pm i0) = \log(-x)\pm i\pi$ for $x<0$.)