18.155, FALL 2021, PROBLEM SET 2

Review / helpful information:

- Some of the exercises below might be solved in the suggested textbooks. You may read these but you need to write your own solutions, and you can only use the statements that are covered in the lecture notes (don't say 'this is Theorem blah in Hörmander, the solution is done').
- Restrictions: if $V \subset U \subset \mathbb{R}^n$ open and $u \in \mathcal{D}'(U)$ then

$$u|_V \in \mathcal{D}'(V), \qquad (u|_V, \varphi) = (u, \varphi) \quad \text{for all} \quad \varphi \in C_c^\infty(V) \subset C_c^\infty(U).$$

- Differentiation: for $u \in \mathcal{D}'(U)$, $\varphi \in C_c^\infty(U)$

$$(\partial_{x_j} u, \varphi) = -(u, \partial_{x_j} \varphi).$$

- Multiplication by smooth functions: for $u \in \mathcal{D}'(U)$, $\varphi \in C_c^\infty(U)$, $a \in C^\infty(U)$

$$(au, \varphi) = (u, a\varphi).$$

1. (Optional) Show that

$$u(\varphi) = \sum_{k=1}^{\infty} \partial_x^k \varphi(1/k), \quad \varphi \in C_c^\infty((0, \infty))$$

defines a distribution on $(0, \infty)$ but this distribution does not extend to $\mathbb{R}$, that is there exists no $v \in \mathcal{D}'(\mathbb{R})$ such that $u = v|_{(0,\infty)}$. (Hint: pair u with a dilated cutoff function whose support contains $1/k$ but no other points of the form $1/j$, $j \in \mathbb{N}$.)

2. (Optional) Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^{n'}$ be open and consider a linear operator

$$A : C_c^\infty(U) \to C_c^\infty(V).$$

Show that the following two definitions of continuity of A are equivalent:

(1) the following two conditions both hold:
   (a) for every compact $K \subset U$ there exists compact $K' \subset V$ such that for all $\varphi \in C_c^\infty(U)$ with $\operatorname{supp} \varphi \subset K$, we have $\operatorname{supp}(A\varphi) \subset K'$ (we can call this 'uniform control on compact support'); and
   (b) for every compact $K \subset U$ and $N \in \mathbb{N}$ there exist $C > 0$, $N' \in \mathbb{N}$ such that we have the seminorm bound

$$\|A\varphi\|_{C^N(V)} \leq C\|\varphi\|_{C^{N'}(U)} \quad \text{for all } \varphi \in C_c^\infty(U) \text{ with } \operatorname{supp} \varphi \subset K;$$

(2) for each sequence $\varphi_k \in C_c^\infty(U)$ such that $\varphi_k \to 0$ in $C_c^\infty(U)$, we have $A\varphi_k \to 0$ in $C_c^\infty(V)$ (this is called 'sequential continuity').

(Hint: for the direction (2) $\Rightarrow$ (1) you can argue by contradiction: if either 1(a) or 1(b) fails then construct a sequence φ_k which violates sequential continuity. In case of 1(a) it helps to take a sequence of compact subsets K_ℓ exhausting V: if 1(a) fails then there exists $K \subset U$ such that neither of the sets K_ℓ will work as K'.)

3. Consider a function $f : \mathbb{R} \to \mathbb{C}$ such that f lies in C^1 on $(-\infty, a)$ and (a, ∞) for some $a \in \mathbb{R}$ and the derivative $f' \in C^0(\mathbb{R} \setminus \{a\})$ is locally integrable on $\mathbb{R}$. The latter implies the existence of one-sided limits $f(a+0)$ and $f(a-0)$. Show that

$$\partial_x f = f' + (f(a+0) - f(a-0))\delta_a$$

where $\partial_x f$ denotes the distributional derivative of $f \in \mathcal{D}'(\mathbb{R})$.

4. Assume that $u, v \in C^0(\mathbb{R})$ and $\partial_x u = v$ in the sense of distributions in $\mathcal{D}'(\mathbb{R})$. Show that $u \in C^1(\mathbb{R})$ and $u' = v$ in the sense of the ordinary derivative. That is, if the distributional derivative is continuous, then it is the ordinary derivative.

5. (a) For $m \in \mathbb{N}$, write $x\partial_x^m \delta_0 \in \mathcal{D}'(\mathbb{R})$ as a linear combination of $\delta_0, \partial_x \delta_0, \ldots, \partial_x^{m-1}\delta_0$.

(b) Show that the space of solutions to the equation $x^m u = 0$, $u \in \mathcal{D}'(\mathbb{R})$, is the span of $\delta_0, \partial_x \delta_0, \ldots, \partial_x^{m-1}\delta_0$. (Hint: for $m = 1$ this was done in class. The $m = 1$ result can be iterated to get the general case.)

6. (a) Assume that $\varphi \in C^\infty(\mathbb{R}^n)$. Show that there exist $\psi_1, \ldots, \psi_n \in C^\infty(\mathbb{R}^n)$ such that

$$\varphi(x) = \varphi(0) + x_1 \psi_1(x) + \cdots + x_n \psi_n(x).$$

(Hint: apply the Fundamental Theorem of Calculus to the function $t \mapsto \varphi(tx)$.)

(b) Show that every solution $u \in \mathcal{D}'(\mathbb{R}^n)$ to the system of equations $x_1 u = \cdots = x_n u = 0$ is a constant multiple of δ_0.

7. (Optional) Find all $u \in \mathcal{D}'(\mathbb{R})$ such that $u \sin x = 0$.