18.155, FALL 2021, PROBLEM SET 11

Review / helpful information:

- If $\mathcal{X}, Y$ are Banach spaces, then a bounded operator $P : \mathcal{X} \to \mathcal{Y}$ is Fredholm if the kernel $\ker P$ is finite dimensional, the range $\operatorname{Ran} P$ is a closed subspace of $\mathcal{Y}$, and $\operatorname{Ran} P$ has finite codimension. The index of P is
$$\operatorname{ind} P := \dim\ker P - \operatorname{codim}\operatorname{Ran} P.$$
If $Q : \mathcal{X} \to \mathcal{Y}$ is a compact operator, then $P+Q$ is Fredholm and $\operatorname{ind}(P+Q) = \operatorname{ind} P$.
- If M is a compact manifold and $P \in \operatorname{Diff}^m(M)$ is an elliptic differential operator, then
$$P_s = P : H^s(M) \to H^{s-m}(M)$$
is a Fredholm operator for each $s \in \mathbb{R}$. The kernel of P_s is independent of s because elements of it are in $C^\infty(M)$ by Elliptic Regularity III; denote this by $\ker P$. We have
$$\operatorname{Ran} P_s = \{w \in H^{s-m}(M) \mid \forall v \in \ker P^* : \langle w, v\rangle_{L^2} = 0\}$$
where $P^* \in \operatorname{Diff}^m(M)$ is the (formal) adjoint of P.

1. Show that the following elliptic estimate for the Laplacian Δ on $\mathbb{R}^2$,
$$\|\psi u\|_{H^2(\mathbb{R}^2)} \le C\|\chi\Delta u\|_{L^2(\mathbb{R}^2)} + C\|\chi u\|_{L^2(\mathbb{R}^2)}$$
does not hold when $\psi = \chi$. (You may choose $\chi \in C^\infty_c(\mathbb{R}^2)$ as you want. Hint: try to construct a sequence of solutions to $\Delta u = 0$ of the form $f(x_1)g(x_2)$.)

2. Assume that (M, g) is a compact connected Riemannian manifold and denote by Δ_g the Laplace–Beltrami operator. Using the material from lecture notes §16 (but not from later sections), show that for any $s \in \mathbb{R}$, the equation
$$\Delta_g u = f, \quad f \in H^{s-2}(M) \quad \text{given},$$
has a solution $u \in H^s(M)$ if and only if $\int_M f\, d\operatorname{vol}_g = 0$.

3. Show that if M is a compact manifold and $P \in \operatorname{Diff}^m(M)$ is an elliptic differential operator, then $P : H^s(M) \to H^{s-m}(M)$ has index 0. (However, differential operators on sections of vector bundles, as well as scalar pseudodifferential operators, can have nonzero index. Hint: first show that $\operatorname{ind}(P) = -\operatorname{ind}(P^t)$ where P^t is the adjoint of P, which has principal symbol $(-1)^m\sigma(P)$. Next show that if two operators in $\operatorname{Diff}^m(M)$ have the same principal symbol, then their index is the same.)

4. (Optional) This exercise gives a basic example of a 0^{th} order pseudodifferential operator on the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ which has nonzero index. Consider the operators $\Pi^\pm$ on $L^2(\mathbb{S}^1)$ defined using Fourier series as follows:

$$\Pi^\pm\left(\sum_{k\in\mathbb{Z}} c_k e^{ikx}\right) = \sum_{\substack{k\in\mathbb{Z}\\ \pm k>0}} c_k e^{ikx}$$

for any sequence $(c_k) \in \ell^2(\mathbb{Z})$. Let $\ell \in \mathbb{Z}$ and define the operator P on $L^2(\mathbb{S}^1)$ by

$$Pf(x) = e^{i\ell x}\Pi^+ f(x) + \Pi^- f(x), \quad f \in L^2(\mathbb{S}^1).$$

Show that P is a Fredholm operator of index $-\ell$. (With more knowledge of microlocal analysis, one could actually show that this is true with $e^{i\ell x}$ replaced by any nonvanishing function $a \in C^\infty(\mathbb{S}^1)$, and the index of P is minus the winding number of the curve $a : \mathbb{S}^1 \to \mathbb{C}$ about the origin – this is a ‘baby index theorem’.)