18.155, FALL 2021, PROBLEM SET 10

Review / helpful information:

- Kohn–Nirenberg symbol class: if $U \subset \mathbb{R}^n$ is open and $\ell \in \mathbb{R}$, then $S^\ell(U\times\mathbb{R}^n) \subset C^\infty(U\times\mathbb{R}^n)$ consists of functions $a(x,\xi)$ such that for each α, β, and a compact set $K \subset U$, there exists $C = C_{\alpha\beta K}$ such that
$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C\langle\xi\rangle^{\ell-|\beta|} \quad \text{for all} \quad x \in K,\ \xi \in \mathbb{R}^n.$$
- If $a \in S^\ell(U\times\mathbb{R}^n)$, then $\mathrm{Op}(a) : C_c^\infty(U) \to C^\infty(U)$ is defined by
$$\mathrm{Op}(a)\varphi(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{\varphi}(\xi)\,d\xi.$$
One sometimes denotes $\mathrm{Op}(a) = a(x, D_x)$, motivated by Exercise 3 below.

1. Show that if $a \in S^\ell(U\times\mathbb{R}^n)$ and $b \in S^r(U\times\mathbb{R}^n)$, then $ab \in S^{\ell+r}(U\times\mathbb{R}^n)$.

2. Show that for any ℓ, the function $\langle\xi\rangle^\ell$ lies in $S^\ell(U\times\mathbb{R}^n)$.

3. Assume that $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ is a polynomial of degree m in ξ with coefficients $a_\alpha(x)$ which are smooth functions on U. Show that $\mathrm{Op}(a)$ is a differential operator:
$$\mathrm{Op}(a)\varphi(x) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha\varphi(x), \quad D_x := -i\partial_x.$$

4. Show that if $a \in S^\ell(U\times\mathbb{R}^n)$, then the transpose $\mathrm{Op}(a)^t : \mathcal{E}'(U) \to \mathcal{D}'(U)$ restricts to a sequentially continuous operator $C_c^\infty(U) \to C^\infty(U)$. (This implies that $\mathrm{Op}(a)$ extends by duality to an operator $\mathcal{E}'(U) \to \mathcal{D}'(U)$. Another way to prove this would be to use Sobolev spaces, but please don't use them in your solution to this exercise.) (Hint: write $\mathrm{Op}(a)^t\varphi = \widehat{B\varphi}$ where B is a certain integral operator. Then show that if $\varphi \in C_c^\infty(U)$ then $B\varphi(\xi) = \mathcal{O}(\langle\xi\rangle^{-\infty})$, either by using Fourier transform or directly by repeated integration by parts.)

5. (Optional) In this exercise you show the following version of Borel's Theorem: for any sequence $a_k \in \mathbb{C}$, $k = 0, 1, \dots$, there exists $f \in C^\infty(\mathbb{R})$ such that $f^{(k)}(0)/k! = a_k$ for all k.

(a) Fix $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ near 0. Show that there exists a sequence $\varepsilon_k > 0$, $k = 0, 1, \dots$, such that $\varepsilon_k \to 0$ and
$$\max_{0\le j<k}\sup_x |\partial_x^j g_k(x)| \le 2^{-k} \quad \text{where} \quad g_k(x) := \chi\Big(\frac{x}{\varepsilon_k}\Big)a_k x^k.$$

(b) Show that the series

$$f(x) := \sum_{k=0}^{\infty} g_k(x)$$

converges in $C^j_{\mathrm{c}}(\mathbb{R})$ for every j to a function $f \in C^\infty_{\mathrm{c}}(\mathbb{R})$ and $f^{(j)}(0)/j! = a_j$ for all j.

6. Show that if $a \in S^\ell(U \times \mathbb{R}^n)$, then $\mathrm{Op}(a)^t$ is a bounded operator $H^s_{\mathrm{c}}(U) \to H^{s-\ell}_{\mathrm{loc}}(U)$ for all s. (Hint: using the mapping properties of $\mathrm{Op}(a)$ on Sobolev spaces, show that for each $\chi \in C^\infty_{\mathrm{c}}(U)$, $u \in H^s(\mathbb{R}^n)$, $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, we have the bound $|(\chi\, \mathrm{Op}(a)^t \chi u, \varphi)| \leq C\|u\|_{H^s(\mathbb{R}^n)}\|\varphi\|_{H^{\ell-s}(\mathbb{R}^n)}$ where the constant C depends on a, χ, s, but not on u or φ. Then use Exercise 1(b) from Problemset 8, together with Continuous Linear Extension. Here the proof of Exercise 1(b) also gives a norm bound – you should state it but don't need to prove it.)