§9. Fundamental solutions

§9.1. Basic properties

Let P be a constant coefficient differential operator on $\mathbb{R}^n$, i.e.

$$P = \sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha$$

for some m & some $a_\alpha \in \mathbb{C}$.

We will study solutions to

$$Pu = f \text{ where } u, f \in \mathcal{D}'(\mathbb{R}^n).$$

The key object will be a fundamental solution:

Defn. We say $E \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of P, if

$$PE = \delta_0.$$

Note: $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$(PE, \varphi) = (E, P^t \varphi)$$

where P^t is the transpose of P:

$$P^t = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} a_\alpha \partial_x^\alpha .$$

For any $u, v \in \mathcal{D}'(\mathbb{R}^n)$ such that supp u, supp v sum properly, we have

$$(Pu) * v = P(u * v) = u * Pv.$$

This gives

Thm Let E be a fundamental soln of P.

① Assume that $u \in \mathcal{D}'(\mathbb{R}^n)$ and supp u, supp E sum properly. Then $u = E * (Pu)$ ("uniqueness of solutions")

② Assume that $f \in \mathcal{D}'(\mathbb{R}^n)$ and supp f, supp E sum properly. Then $P(E * f) = f$ ("existence of solutions")

Remark 1.

The Malgrange – Ehrenpreis Thm says that any P (except $P=0$) has a fundamental solution.

We won't prove it but you can look at [Friedlander–Joshi, §10.4].

Remark 2

The Thm above applies in particular when
1. $u \in \mathcal{E}'(\mathbb{R}^n)$
2. $f \in \mathcal{E}'(\mathbb{R}^n)$.

Together with Remark 1 we see:

1. if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $Pu = 0$ then $u = 0$
2. if $f \in \mathcal{E}'(\mathbb{R}^n)$ then $\exists u \in \mathcal{D}'(\mathbb{R}^n) : Pu = f$

<u>Note:</u> 1 is not uniqueness because being compactly supported is very restrictive.

E.g. $\Delta u = 0$ on $\mathbb{R}^2$ has no nontrivial compactly supported solutions but it has plenty of non-compactly supported ones

e.g. $(x_1 \pm i x_2)^{\ell}$ $\forall \ell \in \mathbb{N}_0$.

Proof of Thm

1. We have
$$E*(Pu) = (PE)*u = \delta_0 * u = u.$$

2. We have
$$P(E*f) = (PE)*f = \delta_0 * f = f. \quad \square$$

§9.2. Examples of fundamental solutions

$\boxed{1}$ Laplace operator:
$$\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2.$$

A fundamental solution is given by the locally integrable function

$$E(x) = \begin{cases} \dfrac{|x|}{2}, & n=1 \\ \dfrac{1}{2\pi}\log|x|, & n=2 \\ \dfrac{c_n}{|x|^{n-2}}, & n>2 \end{cases}$$

where $c_n = -\dfrac{1}{(n-2)\,\mathrm{Volume}(S^{n-1})}$

$S^{n-1} = (n-1)$-dim. sphere in $\mathbb{R}^n$.

<u>Rmk</u>: note E is homogeneous of degree $2-n$ $(n\neq 2)$

So ΔE is homogeneous of degree $-n$, matching δ_0 ...

Proof. Let's just do $n=3$

where $E(x) = -\frac{1}{4\pi|x|}$ (a.k.a. electric potential of point charge)

1 First of all, $\Delta E = 0$

in the classical sense on $\mathbb{R}^3 \setminus \{0\}$

by direct computation: for $x \neq 0$

$$\partial_{x_j} E(x) = \frac{x_j}{4\pi|x|^3}$$

$$\partial^2_{x_j} E(x) = \frac{1}{4\pi|x|^3} - \frac{3x_j^2}{4\pi|x|^5}$$

$$\sum_{j=1}^{3} \partial^2_{x_j} E(x) = 0.$$

2 Now take $\varphi \in C_c^\infty(\mathbb{R}^3)$. We need to show

$(\Delta E, \varphi) = (\delta_0, \varphi) = \varphi(0)$. We have

$$(\Delta E, \varphi) = (E, \Delta\varphi) = \int_{\mathbb{R}^3} E(x)\,\Delta\varphi(x)\,dx$$

$$= \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} E(x)\,\Delta\varphi(x)\,dx.$$

Use Green's 2nd identity on $\mathbb{R}^3 \setminus B(0,\varepsilon)$:

[$\varphi(x) = 0$ for $|x|$ large enough; unit normal to $\partial B(0,\varepsilon)$ is $\nu(x) = -\frac{x}{|x|}$]:

$$\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} \Big(E(x)\Delta\varphi(x) - \underbrace{\varphi(x)\Delta E(x)}_{=0}\Big)dx$$

$$= \underbrace{\int_{\partial B(0,\varepsilon)} E(x)(\nabla\varphi(x)\cdot\nu)dS}_{I_\varepsilon} - \underbrace{\int_{\partial B(0,\varepsilon)} \varphi(x)(\nabla E(x)\cdot\nu)dS}_{II_\varepsilon}$$

$$|I_\varepsilon| \le \int_{\partial B(0,\varepsilon)} \left|\frac{\nabla\varphi(x)\cdot\nu}{4\pi\varepsilon}\right| dS \le \frac{\sup|\nabla\varphi|}{4\pi\varepsilon}\cdot 4\pi\varepsilon^2 \xrightarrow[\varepsilon\to 0+]{} 0.$$

$$II_\varepsilon = -\int_{\partial B(0,\varepsilon)} \varphi(x)\left(\frac{x}{4\pi|x|^3}\cdot\left(-\frac{x}{|x|}\right)\right)dS(x)$$

$$= \int_{\partial B(0,\varepsilon)} \frac{\varphi(x)}{4\pi\varepsilon^2}\,dS(x) = \varphi(0) + O(\varepsilon)$$

Since $\varphi(x) = \varphi(0) + O(\varepsilon)$ when $x \in \partial B(0,\varepsilon)$

So we indeed get $(\Delta E, \varphi) = \varphi(0)$. $\square$

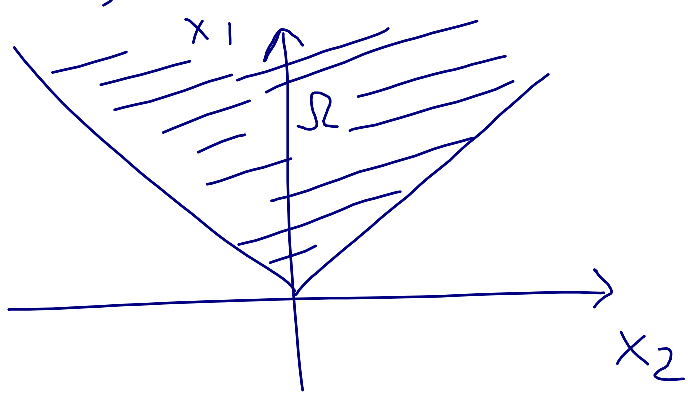
$\boxed{2}$ Wave operator in 1+1 D:

$$P = \partial_{x_1}^2 - \partial_{x_2}^2 \qquad \begin{pmatrix} x_1 = \text{time} \\ x_2 = \text{space} \end{pmatrix}$$

A fundamental solution is given by the locally integrable function

$$E(x_1, x_2) = \begin{cases} \frac{1}{2}, & \text{if } x_1 > |x_2| \\ 0, & \text{if } x_1 < |x_2| \end{cases}$$

Denote
$\Omega = \{x_1 \geq |x_2|\}$
so $\operatorname{supp} E = \Omega$



Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. We need to show $\varphi(0) = (PE, \varphi) = (E, P\varphi)$. Compute

$$(E, P\varphi) = \frac{1}{2}\int_\Omega \partial_{x_1}^2 \varphi - \partial_{x_2}^2 \varphi \, dx_1 dx_2 .$$ Now,

$$\int_\Omega \partial_{x_1}^2 \varphi \, dx_1 dx_2 = \int_{\mathbb{R}} \int_{|x_2|}^{\infty} \partial_{x_1}^2 \varphi(x_1, x_2)\, dx_1 dx_2$$

$$= -\int_{\mathbb{R}} \partial_{x_1} \varphi(|x_2|, x_2)\, dx_2$$

And $\int_{\Omega} \partial_{x_2}^2 \varphi \, dx_1 dx_2 =$

$$= \int_0^\infty \int_{-x_1}^{x_1} \partial_{x_2}^2 \varphi(x_1, x_2) \, dx_2 \, dx_1$$

$$= \int_0^\infty \partial_{x_2} \varphi(x_1, |x_1|) - \partial_{x_2} \varphi(x_1, -|x_1|) dx_1$$

Define $\varphi_+(t) = \varphi(t,t)$ for $t > 0$.
$\varphi_-(t) = \varphi(t,-t)$

Then $(E, P\varphi) =$

$$= \frac{1}{2}\Big(- \int_0^\infty \partial_{x_1} \varphi(t,t) dt - \int_0^\infty \partial_{x_1} \varphi(t,-t) dt$$

$$- \int_0^\infty \partial_{x_2} \varphi(t,t) dt + \int_0^\infty \partial_{x_2} \varphi(t,-t) dt \Big)$$

$$= \frac{1}{2}\Big(- \int_0^\infty \varphi_+'(t) dt - \int_0^\infty \varphi_-'(t) dt \Big)$$

$$= \frac{1}{2}\big(\varphi_+(0) + \varphi_-(0)\big) = \varphi(0,0). \quad \square$$

(In Pset, will derive the d'Alembert's for the solution of the Cauchy problem from this)

§9.3. Singular support

For $u \in \mathcal{D}'(U)$, $U \subset \mathbb{R}^n$ open define the <u>singular support</u> (closed set) $\operatorname{sing\,supp} u \subset U$ as follows:

$x \in U$ does <u>not</u> lie in $\operatorname{sing\,supp} u$ if there exists an open set $V \subset U$, $x \in V$ such that $u|_V \in C^\infty(V)$

(more pedantically, $\exists f \in C^\infty(V)$ s.t. $\forall \varphi \in C_c^\infty(V)$, $(u, \varphi) = \int_V f \varphi \, dx$)

<u>Note</u>: $\operatorname{sing\,supp} u = \emptyset \iff u \in C^\infty(U)$.

Indeed, $\Leftarrow$ is immediate and for $\Rightarrow$, piece together the smooth representations of u on open sets covering u to get that smooth representation on the entire U...

Basic properties:

- $\operatorname{sing\,supp}(\partial_{x_j} u) \subset \operatorname{sing\,supp} u$
- $a \in C^\infty(\overline{U}) \Rightarrow \operatorname{sing\,supp}(au) \subset \operatorname{sing\,supp} u$
- $\operatorname{sing\,supp} u \subset \operatorname{supp} u$
- if $u, v \in \mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{supp} u$, $\operatorname{supp} v$ sum properly then

$$\operatorname{sing\,supp}(u * v) \subset \operatorname{sing\,supp} u + \operatorname{sing\,supp} v.$$

Proof (1) First assume that $u, v \in \mathcal{E}'(\mathbb{R}^n)$.

The idea is to write $u =$ (sth. with controlled supp + C_c^∞ function)

and same for v.

More precisely, take $\psi_u, \psi_v \in C_c^\infty(\mathbb{R}^n)$ such that $\psi_u = 1$ near $\operatorname{sing\,supp} u$

$$\operatorname{supp} \psi_u \subset \operatorname{sing\,supp} u + B(0,\varepsilon)$$

and similarly for ψ_v. (Here $\varepsilon > 0$ arbitrary.)

Now write

$$u = \psi_u u + (1-\psi_u) u$$

where $\text{supp}(\psi_u u) \subset \text{supp}\, \psi_u$

and $(1-\psi_u) u \in C_c^\infty(\mathbb{R}^n)$

Similarly for v. Then

$$u * v = (\psi_u u) * (\psi_v v) + \\ + (\psi_u u) * ((1-\psi_v) v) \\ + ((1-\psi_u) u) * v.$$

The 1st term has

$$\text{sing supp}((\psi_u u) * (\psi_v v)) \subset \text{supp}((\psi_u u) * (\psi_v v)) \\ \subset \text{supp}\, \psi_u + \text{supp}\, \psi_v.$$

The 2nd term is in C_c^∞ as $(1-\psi_v) v \in C_c^\infty$ & similarly the 3rd term is in C_c^∞. So

$$\text{sing supp}(u * v) \subset \text{supp}\, \psi_u + \text{supp}\, \psi_v$$

$$\subset \text{sing supp}\, u + B(0,\varepsilon) + \text{sing supp}\, v + B(0,\varepsilon)$$

$$\subset \text{sing supp}\, u + \text{sing supp}\, v + B(0, 2\varepsilon) \quad \forall \varepsilon > 0$$

Letting $\varepsilon > 0$, get the needed statement.

2 Now we handle the general case supp u, supp v sum properly.

Take any $\chi \in C_c^\infty(\mathbb{R}^n)$.

Then the construction of $u*v$ shows that $\exists\, \tilde{\chi} \in C_c^\infty(\mathbb{R}^n)$:

$$\chi(u*v) = \chi\big((\tilde{\chi}u) * (\tilde{\chi}v)\big).$$

(more precisely, need $\forall (x,y) \in \operatorname{supp} u \times \operatorname{supp} v$ if $x+y \in \operatorname{supp} \chi$ then $\tilde{\chi} = 1$ near x, y).

So by part 1 for $\tilde{\chi}u, \tilde{\chi}v \in \mathcal{E}'(\mathbb{R}^n)$,

$$\operatorname{sing\,supp}(\chi(u*v)) \subset \operatorname{sing\,supp}((\tilde{\chi}u)*(\tilde{\chi}v))$$

$$\subset \operatorname{sing\,supp} u + \operatorname{sing\,supp} v.$$

Since this is true $\forall \chi \in C_c^\infty(\mathbb{R}^n)$,

we get $\operatorname{sing\,supp}(u*v) \subset \operatorname{sing\,supp} u + \operatorname{sing\,supp} v$.

$\square$

§9.4. Elliptic regularity I.

Here we prove ($U \subset \mathbb{R}^n$ open)

Thm Assume that P is a constant coefficient differential operator on $\mathbb{R}^n$ such that it has a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$ with $\operatorname{sing\,supp} E = \{0\}$.
Then we have $\forall u \in \mathcal{D}'(U)$,
$$\operatorname{sing\,supp} u \subset \operatorname{sing\,supp}(Pu).$$
In particular,
$$Pu \in C^\infty(U) \Rightarrow u \in C^\infty(U).$$

Remark The property $\operatorname{sing\,supp} E = \{0\}$ for $P = \Delta$ but not for $P = \partial_{x_1}^2 - \partial_{x_2}^2$.

Proof It suffices to show that
$(\operatorname{sing\,supp} u) \cap K \subset \operatorname{sing\,supp}(Pu)$
for all compact $K \subset U$.
Take a cutoff $\chi \in C_c^\infty(U)$,
$\chi = 1$ near K. Then
$\chi u \in \mathcal{E}'(U)$ and we extend it
by 0 to $\chi u \in \mathcal{E}'(\mathbb{R}^n)$.
Since $PE = \delta_0$ we have
$\chi u = E * (P\chi u)$. Thus
$(\operatorname{sing\,supp} u) \cap K \subset \operatorname{sing\,supp}(\chi u) \subset$
$\subset \operatorname{sing\,supp} E + \operatorname{sing\,supp}(P\chi u)$
$\subset \operatorname{sing\,supp}(P\chi u)$ as $\operatorname{sing\,supp} E = \{0\}$.

Now $P\chi u = \chi Pu + [P, \chi]u$
where $[P, \chi] = P\chi - \chi P$
is a differential operator
with variable coefficients, which
are supported away from K as $\chi = 1$ near K.

We have

$$\operatorname{sing\,supp}(\chi P u) \subset \operatorname{sing\,supp}(Pu)$$

and $\operatorname{sing\,supp}([P,\chi]u) \cap K = \emptyset$

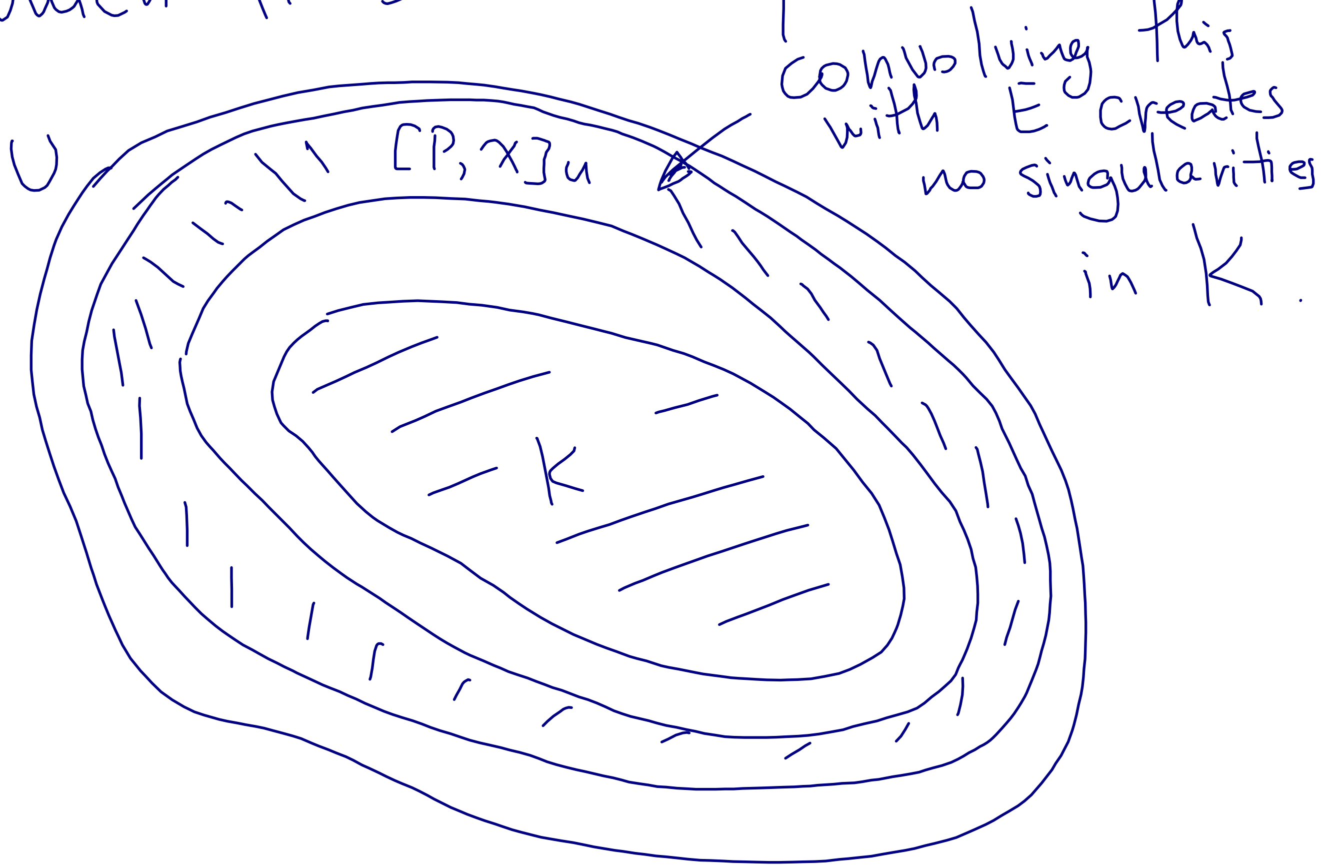
since $\operatorname{supp}([P,\chi]u) \cap K = \emptyset$.

So $(\operatorname{sing\,supp} u) \cap K \subset$

$$\subset (\operatorname{sing\,supp}(\chi P u) \cup \operatorname{sing\,supp}([P,\chi]u)) \cap K$$

$$\subset \operatorname{sing\,supp}(Pu)$$

which finishes the proof. $\square$



Example on convolutions & fund. solutions

(should go right after §9.1)

We know $(\partial^\alpha u) * v = u * \partial^\alpha v$
if supp u, supp v sum properly.
Otherwise this might be false
even if $(\partial^\alpha u) * v$, $u * \partial^\alpha v$ are
well-defined.
In particular, $u = E * Pu$ only
if supp u, E sum properly.

Example: on $\mathbb{R}$, take $P = \partial_x$,
$E = H(x)$, $u = 1$. Then
$(\partial_x E) * u = \delta_0 * u = 1$ but
$E * \partial_x u = E * 0 = 0$.

How does this work? If $\varphi \in C_c^\infty(\mathbb{R})$ then

$$((\partial_x E) * u, \varphi) = (\partial_x E(x) \otimes u(y), \varphi(x+y))$$
$$\text{"="}\ (H(x) \otimes 1(y), \varphi'(x+y)) \quad \text{and}$$
$$(E * \partial_x u, \varphi) = (E(x) \otimes \partial_x u(y), \varphi(x+y))$$
$$\text{"="}\ (H(x) \otimes 1(y), \varphi'(x+y)) \quad \text{as well.}$$

But $(H(x) \otimes 1(y), \varphi'(x+y))$
$= \int_{x>0} \varphi'(x+y)\,dx\,dy$ does not converge.

And Fubini's Thm fails for this integral:

$$\int_0^\infty \left(\int_{-\infty}^\infty \varphi'(x+y)\,dy \right) dx = \int_0^\infty 0\,dx = 0 \quad \text{but}$$

$$\int_{-\infty}^\infty \left(\int_0^\infty \varphi'(x+y)\,dx \right) dy = -\int_{-\infty}^\infty \varphi(y)\,dy.$$