# §8. Convolutions II



## §8.1. Convolution of distributions

Imagine $u, v \in L^1_c(\mathbb{R}^n)$.

Then the convolution is

$$u * v(x) = \int_{\mathbb{R}^n} u(y)\, v(x-y)\, dy.$$

How to view this distributionally?

Take $\varphi \in C_c^\infty(\mathbb{R}^n)$, then

$$(u * v, \varphi) = \int_{\mathbb{R}^n} u * v(x) \cdot \varphi(x)\, dx$$

$$= \int_{\mathbb{R}^{2n}} u(y)\, v(x-y)\, \varphi(x)\, dy\, dx$$

$$\boxed{x \to x+y}$$

$$= \int_{\mathbb{R}^{2n}} u(y)\, v(x)\, \varphi(x+y)\, dx\, dy$$

$$= (u \otimes v, \varphi(x+y)).$$

This leads to

**Defn.** Let $u, v \in \mathcal{E}'(\mathbb{R}^n)$.

Define $u * v \in \mathcal{E}'(\mathbb{R}^n)$ as follows:

$\forall \varphi \in C^\infty(\mathbb{R}^n)$, define

$$(u * v, \varphi) := (u(x) \otimes v(y), \varphi(x+y))$$

where $\varphi(x+y) \in C^\infty(\mathbb{R}^{2n})$.

(Exercise: check $u*v$ is indeed a distribution)

## Basic properties

① $v \in C_c^\infty(\mathbb{R}^n) \Longrightarrow$ get the same convolution as defined in §6.1.

To see this, need to show that

$\forall \varphi \in C^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (u(y), v(x-y))\, \varphi(x)\, dx = (u(y) \otimes v(x), \varphi(x+y))$$

But, using Riemann sums in $x$ as in §6.2,

$$\int_{\mathbb{R}^n} (u(y), v(x-y))\, \varphi(x)\, dx = \left(u(y), \int_{\mathbb{R}^n} v(x-y)\varphi(x)\, dx\right)$$

$$= \left(u(y), \int_{\mathbb{R}^n} v(x)\, \varphi(x+y)\, dx\right)$$

$$= (u(y), (v(x), \varphi(x+y)))$$
$$= (u(y) \otimes v(x), \varphi(x+y)). \quad \Box$$

2 $u * v = v * u$ (since $\varphi(x+y) = \varphi(y+x)$)

3 $u, v, w \in \mathcal{E}'(\mathbb{R}^n) \Rightarrow$
$\Rightarrow \quad u * (v * w) = (u * v) * w$

<u>Proof</u>: if $\varphi \in C_c^\infty(\mathbb{R}^n)$ then

$$(u * (v * w), \varphi)$$
$$= (u(x) \otimes (v * w)(y), \varphi(x+y))$$
$$= (u(x), (v * w(y), \varphi(x+y)))$$
$$= (u(x), (v(y) \otimes w(z), \varphi(x+y+z)))$$
$$= (u(x) \otimes (v(y) \otimes w(z)), \varphi(x+y+z))$$

Get the same for $((u * v) * w, \varphi)$
because $u \otimes (v \otimes w) = (u \otimes v) \otimes w$.

4 $\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v$.

Indeed, if $\varphi \in C^\infty(\mathbb{R}^n)$ and
$\operatorname{supp} \varphi \cap (\operatorname{supp} u + \operatorname{supp} v) = \emptyset$ then $\operatorname{supp}(\varphi(x+y)) \cap$
$\cap \operatorname{supp}(u \otimes v) = \emptyset$.

## 5 Convolution and differentiation:

$$\partial^\alpha(u*v) = (\partial^\alpha u)*v = u*(\partial^\alpha v).$$

Proof Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then

$(\partial^\alpha(u*v), \varphi) = (-1)^{|\alpha|}(u*v, \partial_\alpha \varphi)$

by definition of $\partial^\alpha$ on distributions

$= (-1)^{|\alpha|}(u(x)\otimes v(y), (\partial^\alpha \varphi)(x+y))$

$= (-1)^{|\alpha|}(u(x)\otimes v(y), \partial_x^\alpha(\varphi(x+y)))$

$= (\partial_x^\alpha(u(x)\otimes v(y)), \varphi(x+y))$

$= ((\partial_x^\alpha u)(x)\otimes v(y), \varphi(x+y))$

$= ((\partial^\alpha u)*v, \varphi).$

## 6 Convolution with delta:

$u*\delta_0 = u \quad \forall u \in \mathcal{E}'(\mathbb{R}^n).$

Proof Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then

$(u*\delta_0, \varphi) = (u(x)\otimes\delta_0(y), \varphi(x+y))$

$= (u, \varphi)$ since $(u(x)\otimes\delta_0(y), \beta(x,y))$

$= (u(x), \beta(x,0))$

## §8.2. Convolution without compact support

So far we took $u, v \in \mathcal{E}'(\mathbb{R}^n)$ and defined $u * v \in \mathcal{E}'(\mathbb{R}^n)$.
But what if one or both of $u, v$ are not compactly supported?

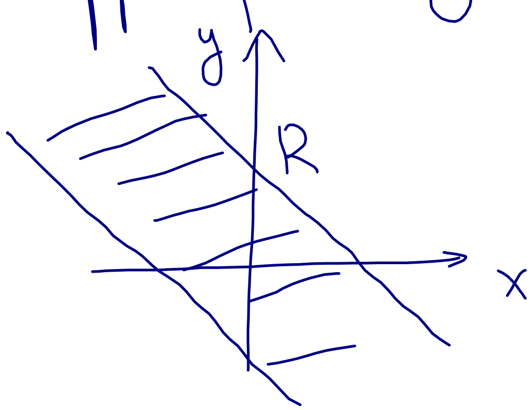
Let $u, v \in \mathcal{D}'(\mathbb{R})$, $\varphi \in C_c^\infty(\mathbb{R})$.
We want to still define

$$(u * v, \varphi) = (u(x) \otimes v(y), \varphi(x+y)).$$

But alas, $(x,y) \mapsto \varphi(x+y)$ does not lie in $C_c^\infty(\mathbb{R}^{2n})$.

Instead, if $R$ is large enough so that $\operatorname{supp} \varphi \subset B(0,R)$, then

$$\operatorname{supp}(\varphi(x+y)) \subset \{(x,y) \in \mathbb{R}^{2n} : |x+y| \leq R\}$$




And $\operatorname{supp}(u \otimes v) =$
$= \operatorname{supp} u \times \operatorname{supp} v$
$= \{(x,y) : x \in \operatorname{supp} u,\ y \in \operatorname{supp} v\}.$

We want

$(\operatorname{supp}\varphi(x+y)) \cap (\operatorname{supp} u \times \operatorname{supp} v)$

to be compact, so that we could define $(u \otimes v, \varphi(x+y))$ by wedging in a cutoff (informally: the "integral" $\int_{\mathbb{R}^{2n}} u(x)v(y)\varphi(x+y)\,dx\,dy$ "converges" because it's over a finite set)

For that we need:

$$x \in \operatorname{supp} u,\ y \in \operatorname{supp} v,\ |x+y| \leq R$$
$$\Downarrow$$
$$\max(|x|, |y|) \leq T(R)$$

<u>Defn.</u> Let $V_1, V_2 \subset \mathbb{R}^n$ be closed. Then we say $V_1, V_2$ <u>sum properly</u> if $\forall R\ \exists T: \forall (x,y) \in V_1 \times V_2$

$$|x+y| \leq R \Rightarrow |x|, |y| \leq T.$$

<u>Note:</u> in this case $V_1 + V_2 \subset \mathbb{R}^n$ is closed (exercise...)

We will now define
$u * v \in \mathcal{D}'(\mathbb{R}^n)$
for $u, v \in \mathcal{D}'(\mathbb{R}^n)$ such that
supp $u$, supp $v$ sum properly.
Take $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then
$\exists R$ : supp $\varphi \subset B(0,R)$.
Thus $\exists T$: if $(x,y) \in \operatorname{supp} u \times \operatorname{supp} v$
and $|x+y| \leq R$, then $|x|, |y| \leq T$.
Fix a cutoff $\chi \in C_c^\infty(\mathbb{R}^n)$,
$\chi = 1$ near $\overline{B(0,T)}$
and define
$$(u * v, \varphi) := ((\chi u) * (\chi v), \varphi)$$
where $\chi u, \chi v \in \mathcal{E}'(\mathbb{R}^n)$, so
their convolution is well-defined:
$$(u * v, \varphi) = (\chi u \otimes \chi v, \varphi(x+y))$$
$$= (u(x) \otimes v(y), \chi(x)\chi(y)\varphi(x+y))$$

• Independence of the choice of $\chi$:

imagine $\chi' \in C_c^\infty(\mathbb{R}^n)$,

$\chi' = 1$ near $\overline{B(0,T)}$.

We want $(\chi u \otimes \chi v, \varphi(x+y)) =$

$= (\chi' u \otimes \chi' v, \varphi(x+y))$. $(*)$

But $\operatorname{supp}(\chi u \otimes \chi v - \chi' u \otimes \chi' v) \subset$

$\subset (\operatorname{supp}((\chi - \chi')u) \times \operatorname{supp} v)$

$\cup (\operatorname{supp} u \times \operatorname{supp}((\chi - \chi')v)$

(writing $\chi u \otimes \chi v - \chi' u \otimes \chi' v$

$= \chi u \otimes (\chi - \chi')v + (\chi - \chi')u \otimes \chi' v)$

So $\boxed{\operatorname{supp}(\chi u \otimes \chi v - \chi' u \otimes \chi' v) \cap \operatorname{supp} \varphi(x+y) = \emptyset}$:

if $(x,y) \in$ this intersection then

$x \in \operatorname{supp} u$, $y \in \operatorname{supp} v$, $|x+y| \leq R$

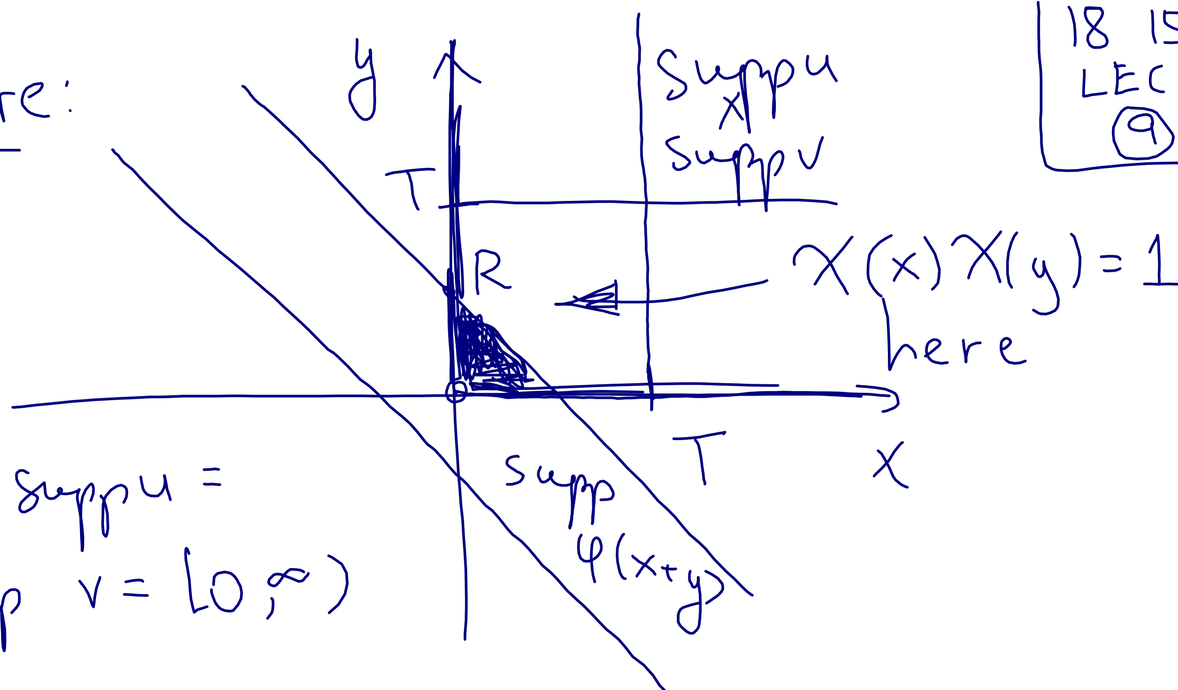
So $|x|, |y| \leq T$. But also

either $x$ or $y$ lies in $\operatorname{supp}(\chi - \chi')$,

a contradiction.

Now we see that $(*)$ holds

Picture:



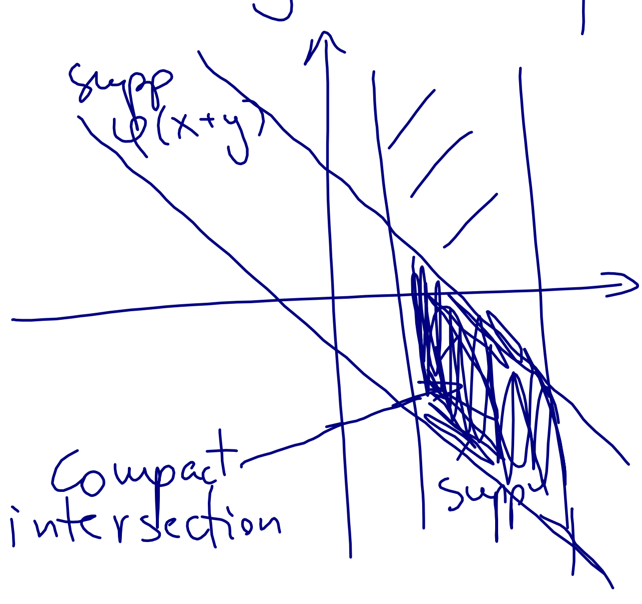

Say $\operatorname{supp} u = \operatorname{supp} v = [0,\infty)$

Now can check that $\varphi \mapsto (u * v, \varphi)$ is linear & continuous in $\varphi$, defining $u * v \in \mathcal{D}'(\mathbb{R}^n)$.

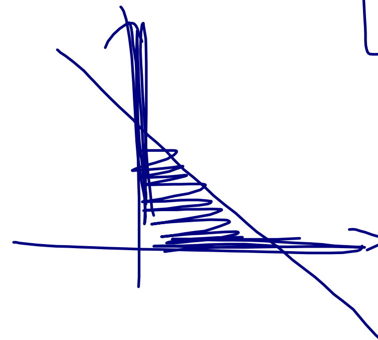
Examples

(1) supp u or supp v compact: supp u, supp v always sum properly

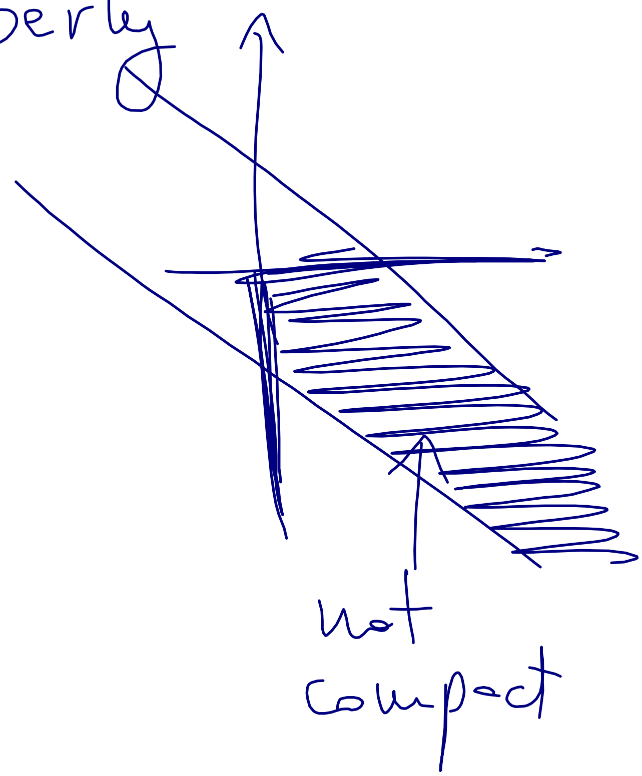
So, can define $u * v \in \mathcal{D}'(\mathbb{R}^n)$ for $u \in \mathcal{D}'(\mathbb{R}^n)$, $v \in \mathcal{E}'(\mathbb{R}^n)$



(2) supp $u$, supp $v \subset [0,\infty)$ CR: again sum properly



(3) supp $u = [0,\infty)$,
supp $v = (-\infty, 0]$
do not sum properly




E.g. for

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$H * H(x) = x_+^1$$

but $H * H_-$ does not make sense where

$$H_-(x) = H(-x).$$

# Properties of convolution:

Still hold but need supports to sum properly.

For property 3 $\left(u*(v*w) = (u*v)*w\right)$

need $u, v, w \in \mathcal{D}'(\mathbb{R}^n)$ and

$\forall R \; \exists T : \; \forall (x,y,z) \in \operatorname{supp} u \times \operatorname{supp} v \times \operatorname{supp} w$

$$|x+y+z| \leq R \Rightarrow |x|, |y|, |z| \leq T.$$